RHOTRIX THEORY

Dissertion submitted in the partial fulfillment of the requirement for the

MASTER'S DEGREE IN MATHEMATICS

SUBMITTED BY

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DECLARATION

I, Rintu Francis hereby declare that this project entitled '**RHOTRIX THEORY**'is a bonifide record of work done by me under the guidance of Dr. Anntreesa Josy, Assistant Professor on contract, Department of Mathematics, Bharata Mata College, Thrikkakara and this work has not previously formed by the basis for the award of any academic qualification, fellowship or other similar title of any other University or Board.

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CERTIFICATE

This is to certify that the project entitled '**RHOTRIX THEORY'** submitted for the partial fulfillment requirement of Master's Degree in Mathematics is the original work done by Rintu Francis during the period of the study in the Department of Mathematics, Bharata Mata College, Thrikkakara under my guidance and has not been included in any other project submitted previously for the award of any degree.

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ABSTRACT

Rhotrix theory is a mathematical framework that extends the classical notion of matrices to a more generalized form, incorporating additional algebraic structures and properties. Rhotrices, the fundamental objects in this theory, are defined to handle operations that traditional matrices cannot efficiently represent, offering a broader applicability in various mathematical and engineering fields. This theory provides new avenues for matrix analysis, facilitating the exploration of higher-dimensional algebraic systems, and can be applied in areas such as cryptography, signal processing, and control theory. The development of rhotrix theory involves the study of its unique operations, properties, and potential applications, aiming to enrich the toolbox available for solving complex mathematical problems.

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INTRODUCTION

Rhotrix is a new topic of study in linear mathematical algebra that focuses on representing an array of integers in rhomboidal shape. A rhotrix A of dimension three is a rhomboidal array defined as,

$$R_3 = \left\langle \begin{array}{ccc} & x & \\ y & z & p \\ & q & \end{array} \right\rangle$$

Rhotrix is still under the beginning stages of development as a concept. Since the concept's inception in 2003, a number of researchers have expressed interest in advancing and developing it. Typically, they do this by establishing comparisons between the notions of matrices and this transformation, which turns a matrix into a rhotrix.In this sense, rhotrices emerged as a new frame work in matrix theory, drawing attention from scholars due to its richer mathematical foundation.

More than forty studies on the Rhotrix idea have been published in the literature . It is important to note that there are now two methods in the literature for multiplying rhotrices.The first is Ajibade's heart-based approach for multiplying rhotrix.The second one were created by B. Sani and A. Mohammed using the row-column based method for rhotrix multiplication, which divided the 40 articles in the rhotrix theory literature. In this area, B. Sani made another contribution in by providing row-column multiplication of higher dimensional rhotrices in, expanding on previous work on row-column multiplication for dimension three base level rhotrices and a unique matrix known as a "coupled matrix." This project report consists of three chapters. First chapter includes the basic definition in rhotrix theory. There we define the basic structure and representation of an arbitrary rhotrix and operations on rhotrix. Both of the rhotrice multiplication techniques that were discussed in the preceding paragraphs are defined there. Additionally, we investigate how to transform a rhotrix into a unique matrix known as a coupled matrix. The rank of the rhotrix is covered in the second chapter. The natural rhotrix, determinant and co-determinant functions, index of rhotrices are discussed in last chapter

Chapter 1

SOME FACTS OF RHOTRIX

1.1 Forms of Rhotrix

The terms "axes," "heart," "vertex," and "rows and columns of a rhotrix" are defined in this section.

Definition : 1.1. Axes of a Rhotrix: A rhotrix have two axes. The two axes are the horizontal and vertical axes. An array of entries that extends from the top to the bottom of a rhotrix forms its vertical axis. On the other hand, a rhotrix's horizontal axis is an array of items that extends from the left to the right. Each rhotrix has two primary axes: one horizontal and one vertical.

Example : 1.1. $R_3 = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle.$

Here vertical axis of the rhotrix R_3 is the set of values "x, y, z and the horizontal axis is the set of values q, y, p.

Definition : 1.2. Heart of a Rhotrix: The point where the horizontal and vertical axis intersect is called its Heart, denoted by $h(R)$, where R is the given rhotrix.

Example : 1.2. $\mathbf{R_3} = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$

In this case, element y becomes the heart of the rhotrix R_3. The symbol for it is $h(R_3)$.

Definition : 1.3. Vertex of a Rhotrix: An entry at any of a rhotrix's corners is considered as its vertex.

Example : 1.3. For example

$$R_5 = \left\langle \begin{array}{ccccc} & & x & & \\ & q & p & y & \\ t & u & v & w & z \\ & k & l & m & \\ & & o & & \end{array} \right\rangle$$

Here x, t, o and z. are the vertices of rhotrix

Definition : 1.4. Rows and Columns of a Rhotrix: An array of entries running from the top-left to the bottom-right of a rhotrix is called a row. An array of elements running from the top-right to the left-bottom side of a rhotrix is called a column.

Example : 1.4. For example

$$R_5 = \left\langle \begin{array}{ccccc} & & x & & \\ & q & p & y & \\ t & u & v & w & z \\ & k & l & m & \\ & & o & & \end{array} \right\rangle$$

The first rows is xyz, second row is pw and so on and the first column is xqt, the second column is pu and so on.

1.2 Higher Dimensional Rhotrix

Definition : 1.5. Dimension of a Rhotrix: A rhotrix's dimension is the number of entries along one of its major vertical or horizontal axes.. All rhotrices are of odd dimension (≥ 3).. Here $t = \frac{n^2+1}{2}$. indicates the cardinality or number of elements of a rhotrix of dimension n, where $n \in \mathbb{Z}^+$.

Example : 1.5. For example

$$\mathrm{R}_5 = \left\langle \begin{array}{ccccc} & & x & & \\ & q & p & y & \\ t & u & v & w & z \\ & k & l & m & \\ & & o & & \end{array} \right\rangle$$

R_5 has dimension 5

1.3 Operation Of Rhotrices

1.3.1 Addition of Rhotrix

Only two rhotrices have the same dimension can they be added together. The sum of the matching elements of two rhotrices is the definition of their addition. Let R_3 and Q_3 be two 3-dimensional rhotrices such that,

$$R_3 = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle \text{ and } \mathbf{Q_3} = \left\langle \begin{array}{ccc} & u & \\ s & v & w \\ & t & \end{array} \right\rangle$$

Then their addition is defined as

$$R_3 + Q_3 = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle . + \left\langle \begin{array}{ccc} & u & \\ s & v & w \\ & t & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & x+u & \\ s+q & v+y & p+w \\ & t+z & \end{array} \right\rangle$$

1.3.2 Scalar Multiplication of Rhotrix

In scalar multiplication, the given scalar is multiplied by each entry in a rhotrix.

Let $R_3 = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$ and α be a scalar number. Then the scalar multiplication of a rhotrix is defined as

$$\alpha(\mathbf{R_3}) = \alpha \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & \alpha \mathrm{x} & \\ \alpha \mathrm{q} & \alpha \mathrm{y} & \alpha \mathrm{p} \\ & \alpha \mathrm{z} & \end{array} \right\rangle$$

1.3.3 Multiplication of Rhotrix

Rhotrices can be multiplied in two different ways. Row-column multiplication and heart-oriented multiplication are these. While row-column multiplication resembles matrix multiplication, heart-oriented multiplication is connected to the heart of a rhotrix.

Heart Oriented Multiplication

As the name implies, we multiply each element of the first rhotrix by the heart of the second rhotrix, and we add what comes out to the product of the corresponding element of the second rhotrix and the heart of the first rhotrix.

Let $R_3 = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$ and $Q_3 = \left\langle \begin{array}{ccc} & u & \\ s & v & w \\ & t & \end{array} \right\rangle$

be any two rhotrices, then their heart oriented multiplication is defined as

$$R_3 o Q_3 = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle \text{o} \left\langle \begin{array}{ccc} & u & \\ s & v & w \\ & t & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & xv + uy & \\ qv + sy & yv & pv + wy \\ & zv + ty & \end{array} \right\rangle$$

Remark : 1.1. When two non-zero rhotrices are multiplied using heart-oriented multiplication, the outcome is not always a non-zero rhotrix.

Proof. Let $R_3 = \left\langle \begin{array}{ccc} & x & \\ q & 0 & p \\ & z & \end{array} \right\rangle$ and $Q_3 = \left\langle \begin{array}{ccc} & u & \\ s & 0 & w \\ & t & \end{array} \right\rangle$

Let R and Q be two non-zero rhotrices, then from above result, we have

$$\left\langle \begin{array}{ccc} & x & \\ q & 0 & p \\ & z & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & u & \\ s & 0 & w \\ & t & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle$$

□

Identity Rhotrix for Heart-Oriented Multiplication

The definition of the 3-dimensional identity rhotrix is

$$I_3 = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

Here, I_3 is derived as follows. Let $I_3 = \left\langle \begin{array}{ccc} & m & \\ n & o & d \\ & r & \end{array} \right\rangle$ be the identity rhotrix and $\mathbf{R_3} = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$ be a rhotrix, where $h(R_n) \neq 0$

Since

$$R_3 o I_3 = I_3 o R_3 = R_3$$

we have

$$\left\langle \begin{array}{ccc} & m & \\ r & o & f \\ & n & \end{array} \right\rangle \text{o} \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & my + xo & \\ qo + ry & oy & fy + po \\ & zo + ny & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$$

According to the definition of rhotrice equality, we have

$$my + xo = x$$

$$qo + ry = q$$

$$oy = y$$

$$fy + po = p$$

$$zo + ny = z$$

By solving the above equations we get

$$m = n = d = r = 0 \text{ and } o = 1$$

Therefore, we obtain

$$I_3 = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

Rhotrix Inverse in Heart-Oriented Multiplication

Let $h(R) \neq 0$. and R be a 3-dimensional rhotrix. If a rhotrix P exist such that

$$RoQ = PoR = I$$

then P is referred to as R inverse. Now, we can get a rhotrix's inverse by doing the following:

Let $R = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$ be a 3 dimensional rhotrix such that $y \neq 0$

If P $= \left\langle \begin{array}{ccc} & m & \\ r & o & f \\ & n & \end{array} \right\rangle$ is the inverse of P such that

$$\left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle \text{o} \left\langle \begin{array}{ccc} & m & \\ r & o & f \\ & n & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

$$\left\langle \begin{array}{ccc} & my+xo & \\ qo+ry & oy & fy+po \\ & zo+ny & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

By definition of equality of rhotrices, we get

$$my + xo = 0$$

$$qo + ry = 0$$

$$ry = 1$$

$$fr + po = 0$$

$$zo + ny = 0$$

It follows from that

$$o = \tfrac{1}{y}, m = \tfrac{-x}{y^2}, qr = \tfrac{-q}{y^2}, s = \tfrac{-p}{y^2} \text{ and } t = \tfrac{-z}{y^2}$$

Therefore we have

$$R = P^{-1} = \tfrac{-1}{y^2} \left\langle \begin{array}{ccc} & x & \\ q & -y & p \\ & z & \end{array} \right\rangle$$

Remark 1. The inverse of a unit heart rhotrix

$$\left\langle \begin{array}{ccc} & x & \\ q & 1 & p \\ & z & \end{array} \right\rangle$$

where heart is unity is given by

$$\left\langle \begin{array}{ccc} & -x & \\ -q & 1 & -p \\ & -z & \end{array} \right\rangle$$

A rhotrix If h(R) = 0, then R is invertible.

Proof. If R is invertible then there exist a rhotrix P such that $RoP = I$.

$$h(RoP) = h(I)$$

$$\text{then } h(R)h(P) = 1$$

$$h(P) = \frac{1}{h(R)}$$

$$h(R) \neq 0.$$

Remark : 1.2. Heart-oriented rhotrix multiplication is a group with respect to the set of all invertible 3-dimensional rhotrices over R.

Proof. Let $Q = \left\{ R = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle ; y \neq 0, x, q, p, z \in \mathbb{N} \right\}$

Let R = $\left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$ and S = $\left\langle \begin{array}{ccc} & m & \\ q & n & p \\ & o & \end{array} \right\rangle$ be two elements in Q.

$$\text{Then RoQ} = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle \text{o} \left\langle \begin{array}{ccc} & m & \\ u & n & v \\ & o & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & xn+my & \\ qn+uy & yn & pn+vy \\ & zn+oy & \end{array} \right\rangle$$

It is evident that the values of y, n, and yn differ from zero. As a result, under heart-oriented multiplication, the set Q is closed.

Again for any $R, S, A \in S$ we have

$$\text{A} = \left\langle \begin{array}{ccc} & d & \\ e & f & g \\ & h & \end{array} \right\rangle$$

$$Ro(SoA) = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle \text{o} \left\{ \left\langle \begin{array}{ccc} & m & \\ u & n & v \\ & o & \end{array} \right\rangle \text{o} \left\langle \begin{array}{ccc} & d & \\ e & f & g \\ & h & \end{array} \right\rangle \right\}$$

$$= \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle \text{o} \left\langle \begin{array}{ccc} & mydn & \\ uy+qn & yn & vy+gn \\ & oy+zn & \end{array} \right\rangle$$

$$= (RoS)oP$$

Thus, in the set Q, the heart-oriented multiplication operation is associative. Also,

$$I=\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

is the identity an element of S. Additionally, the fact that each element of S is invertible suggests that the set S is a group under the multiplication with a heart orientation.

Row-Column Multiplication

B. Sani discussed about row-column multiplication of rhotrices, which is an alternate technique for multiplying rhotrices. By multiplying each row of the first rhotrix by each column of the second rhotrix, each element in this approach is obtained.

Let $R_3=\left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$ and $Q_3=\left\langle \begin{array}{ccc} & l & \\ o & m & p \\ & n & \end{array} \right\rangle$ be two rhotrices.

Then the row-column multiplication of rhotrices R_3 and Q_3 is given by,

$$R_3 o Q_3=\left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle \mathrm{o} \left\langle \begin{array}{ccc} & l & \\ o & m & t \\ & n & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & xl+po & \\ ql+zo & my & xt+pn \\ & qt+zn & \end{array} \right\rangle$$

Identity Rhotrix Under Row-Column Multiplication

Let $I_3 = \left\langle \begin{array}{ccc} & l & \\ o & m & t \\ & n & \end{array} \right\rangle$

be the identity rhotrix under multiplication defined. Then, for any rhotrix R_3 we must have

$$R_3 o I_3 = I_3 o R_3 = R_3$$

Let $R_3 = \left\langle \begin{array}{ccc} & e & \\ f & g & h \\ & i & \end{array} \right\rangle$, $c \neq 0$ Then we have

$$\left\langle \begin{array}{ccc} & e & \\ f & g & h \\ & i & \end{array} \right\rangle \mathrm{o} \left\langle \begin{array}{ccc} & l & \\ o & m & t \\ & n & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & e & \\ f & g & h \\ & i & \end{array} \right\rangle$$

This gives,

$$\left\langle \begin{array}{ccc} & el+ho & \\ fl+io & gm & et+fn \\ & ft+in & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & e & \\ f & g & h \\ & i & \end{array} \right\rangle$$

According to the definition of rhotrice equality, we obtain

$$el + ho = e$$

$$fl + io = f$$

$$gm = g$$

$$et + fn = h$$

$$ft + in = i$$

It derives from above that $l = o = n = 0, o = t = 0$ provided $g(ei - fh) \neq 0$.

Hence,

$$I_3 = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle$$

Inverse of a Rhotrix Under Row-Column Multipilication

The inverse of a rhotrix Q_3 is referred to as such if

$$R_3 o Q_3 = Q_3 o R_3 = I_3$$

Let $R_3 = \left\langle \begin{array}{ccc} & x & \\ y & z & k \\ & l & \end{array} \right\rangle$ and let $P_3 = \left\langle \begin{array}{ccc} & f & \\ g & h & i \\ & m & \end{array} \right\rangle$ be the inverse, then $\left\langle \begin{array}{ccc} & x & \\ y & z & k \\ & l & \end{array} \right\rangle \text{o} \left\langle \begin{array}{ccc} & f & \\ g & h & i \\ & m & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle$

Therefore, $\left\langle \begin{array}{ccc} & xf+kg & \\ yf+lg & zh & xi+km \\ & yi+lm & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle$

According to the definition of rhotrice equality, we obtain

$$xf + kg = 1$$

$$yf + lg = 0$$

$$zh = 1$$

$$xi + km = 0$$

$$yi + lm = 1$$

It follows from above that

$$f = \tfrac{l}{xl-yk}, g = \tfrac{-y}{xl-yk}, h = \tfrac{1}{c}, i = \tfrac{-k}{xl-yk}, k = \tfrac{x}{xl-yk}.$$

Therefore, $Q_3 = R_3^{-1} = \frac{1}{xl-yk} \left\langle \begin{array}{ccc} & l & \\ -y & \frac{xl-yk}{c} & -k \\ & x & \end{array} \right\rangle$

provided $c(xl - yk) \neq 0$.

1.4 Coupled Matrix

A new matrix known as the coupled matrix was introduced by B. Sani in order to address the question of transforming rhotrices into matrices and vice versa. This innovative matrix has proven to be a valuable tool in the field of rhotrix theory, enabling researchers to establish connections between the properties of rhotrices and matrices.

Definition : 1.6. Determinant of a Rhotrix: A three-dimensional rhotrix's determinant was defined by B. Sani.

$$det(R_3) = c(ae - bd)$$

Definition : 1.7. Transpose of a Rhotrix: The transposition of a given rhotrix R_n is a rhotrix that is obta0ined by changing the row and column of the rhotrix R_n; it is represented by R_n^T.

$$\text{For example } \mathbf{R_5} = \left\langle \begin{array}{ccccc} & & \mathrm{x}_1 & & \\ & \mathrm{x}_2 & \mathrm{x}_3 & x_4 & \\ x_5 & x_6 & x_7 & x_8 & x_9 \\ & y_1 & y_2 & y_3 & \\ & & y_4 & & \end{array} \right\rangle$$

$$\mathbf{R_5^T} = \left\langle \begin{array}{ccccc} & & \mathrm{x}_1 & & \\ & \mathrm{x}_4 & \mathrm{x}_3 & x_2 & \\ x_9 & x_8 & x_7 & x_6 & x_5 \\ & y_1 & y_2 & y_3 & \\ & & y_4 & & \end{array} \right\rangle$$

where the entires in the horizontal diagonal arranged its reverse order and entries of the vertical diagonal remain unchanged.

Coupled Matrix: Any rhotrix or matrix can be transposed by turning its columns in an anticlockwise direction at a right angle. Similarly, a coupled matrix is created when the columns of a rhotrix are rotated by 45°, which is represented by half transposition $\frac{T}{2}$.

Example : 1.6. For a 5-dimensional rhotrix R_5,we have

$$R_5 = \left\langle \begin{array}{ccccc} & & x_{11} & & \\ & x_{21} & y_{11} & x_{13} & \\ x_{31} & y_{12} & x_{22} & y_{21} & y_{13} \\ & x_{32} & y_{22} & c_{23} & \\ & & x_{33} & & \end{array} \right\rangle$$

$$\mathrm{R}_5^{\frac{T}{2}} = \begin{pmatrix} x_{11} & & x_{12} & & x_{13} \\ & y_{11} & & y_{12} & \\ x_{21} & & x_{22} & & x_{23} \\ & y_{21} & & y_{22} & \\ x_{31} & & x_{32} & & x_{33} \end{pmatrix}$$

The complementary component matrix of the couple matrix $[AC]_5$ is denoted by C in this instance $\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{32} & x_{32} & x_{34} \end{pmatrix}$ and the minor matrix is $\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$
An $n \times n$ square matrix is obtained by adding zeros to the missing locations in a coupled matrix. Therefore, upon completion, the coupled matrix $[AC]_5$ above becomes

$$\begin{pmatrix} x_{11} & 0 & x_{12} & 0 & x_{13} \\ 0 & y_{11} & 0 & y_{12} & 0 \\ x_{21} & 0 & x_{22} & 0 & x_{23} \\ 0 & y_{21} & 0 & y_{22} & 0 \\ x_{31} & 0 & x_{32} & 0 & x_{33} \end{pmatrix}$$

a 5×5 matrix.[San07]

Chapter 2

RANK OF RHOTRIX

In this chapter, we examine the rank of a rhotrix and outline its characteristics, expanding upon concepts within rhotrix theory. Additionally, we detail the conditions that are both necessary and sufficient for a linear map to be depicted over a rhotrix.

2.1 Major and Minor Rhotrices

In order to analyze the rank of a rhotrix and its associated properties, it is essential to establish some fundamental definitions. Therefore, we will begin by examining these definitions before the study of rhotrix's rank.

Definition : 2.1. Let R_n be an n-dimensional rhotrix, defined as $< a_{ij}, c_{kl} >$. Then, the (i, j) elements (a_{ij}) are referred to as the main entries of R_n, while the (k, l) entries (c_{kl}) are referred to as the minor entries of R_n.

Definition : 2.2. The major and minor matrices of a rhotrix $R_n =< a_{ij}, c_{kl} >$ of $n-$dimension are represented by the coupled matrices (a_{ij}) and (c_{kl}). Therefore, the major and minor matrices of R_n are, respectively, (a_{ij}) and (c_{kl}).

Example For example $\mathbf{R_5} = \left\langle \begin{array}{ccccc} & & x & & \\ & y & z & l & \\ k & i & g & e & c \\ & l & j & h & \\ & & m & & \end{array} \right\rangle$

Here the major matrix corresponding to this rhotrix is
$\begin{pmatrix} x & l & c \\ y & g & h \\ k & l & m \end{pmatrix}$ and the minor matrix is $\begin{pmatrix} z & e \\ i & j \end{pmatrix}$

Definition : 2.3. If $(b_ij) = 0$ for every i, j whose sum $i+j$ is odd, then for each odd integer n, a $n \times n$matrix (b_ij) is termed a filled coupled matrix of a rhotrix of dimension n. These entries will be referred to as the filled coupled matrix's null entries.

Example : 2.1. For example the filled coupled matrix of R_5 is given by

$$\begin{pmatrix} x & 0 & l & 0 & c \\ 0 & z & 0 & e & 0 \\ y & 0 & g & 0 & h \\ 0 & i & 0 & j & 0 \\ k & 0 & l & m & 0 \end{pmatrix}$$

Theorem : 2.1. The set of all $n \times n$ filled coupled matrices over F and the set of all $n-$dimensional rhotrices (where n is odd) over F have a one-to-one connection.

Proof. The set of all $n-$dimensional rhotrices is defined as follows: for any $n-$dimensional rhotrix where n is odd, there exists a $n \times n$ matrix (b_{ij}), also known as a filled coupled matrix, such that for any i, j whose sum $i + j$ is odd, $b_{ij} = 0$.
A $n-$dimensional rhotrix exists when we examine a $n \times n$ filled couple matrix over F. Thus, the proof can be found by establishing a one-to-one relationship between the set of all $n-$dimensional rhotrices over F and d matrices over F.

Definition : 2.4. Assume that $R_n =< a_{ij}, c_{kl} >$R's main diagonal was generated by the entires $a_{rr}(1 \leq s \leq t)$ and $c_{ss}(1 \leq r \leq t - 1)$ in the major and minor matrices' respective main diagonals. A right (left) triangular rhotrix is created when all of the entries on the left (right) side of the principal diagonal in are zeros. Trivially, we obtain the following lemma.[MB+14]

Lemma : 2.5. Let $R_n =< a_{ij}, c_{kl} >$ be a left (right) triangular rhotrix if and only if (a_{ij}) and $(c_{kl}$ are lower (upper) triangular matrices together.

Proof. The coupled matrix is obtained by rotating the rhotrix 45° in an anticlockwise manner. The rhotrix is a left(right) triangular rhotrix if the major and minor matrices are lower(upper) triangular matrices, and vice versa, based on the major and minor matrices that were derived from the coupled matrix.[MB+14]

Based on this lemma, it is possible to transform any n-dimensional rhotrix R into a right triangular rhotrix by converting its major and minor matrix into row echelon form through the application of elementary row operations.

Example : 2.2. For example

$$\mathbf{R_5} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 2 & 1 & 1 & \\ 1 & 2 & 2 & 2 & 1 \\ & 1 & 3 & 1 & \\ & & 1 & & \end{array} \right\rangle$$

The major and minor matrix of the rhotrix A is given by B = $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ C = $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ respectively

Now that B and C have been reduced to the row reduced echelon form $(rref)$, we have rref(B) = $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and rref(C) = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

At this point, we obtain a rhotrix by joining these two row-reduced matrices: $\mathbf{R_5} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 1 & \\ & & 0 & & \end{array} \right\rangle$

The rhotrix above is obviously a right triangular rhotrix.

2.2 Rank of Rhotrix

Definition : 2.6. Let an n-dimensional rhotrix $R_n =< aij, ckl >$ be defined.Next, we define rank(R_n), which is the rhotrix R_n's rank, as follows:

$$rank(R_n) = rank(a_{ij}) + rank(c_{kl}),$$

where (c_{kl}) and (a_{ij}) stand for R_n's major and minor matrices, respectively.

Remember that the number of non-zero rows in a matrix's row reduced echelon form equals its rank.

Example : 2.3. Let

$$\mathbf{R_5} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 2 & -2 & \\ 1 & -1 & 3 & 1 & 2 \\ & -2 & 1 & 1 & \\ & & 1 & & \end{array} \right\rangle$$

Next, A's filled coupled matrix is provided by

m(A) = $\begin{pmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 2 \end{pmatrix}$ major B = $\begin{pmatrix} 1 & -2 & 2 \\ 0 & 3 & 1 \\ 1 & -2 & 2 \end{pmatrix}$ and the minor matrix is $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$

reduce echelon matrix of B

rref(B) $\begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{8}{3} \\ 0 & 0 & 0 \end{pmatrix}$

Note that there are exactly two non-zero rows in $rref(B)$. Consequently, $rank(B) = 2$. Given that C has a non-zero determinant, $rank(C) = 2$. Consequently, $rank(A) = 4$ since by definition, $rank(A) = rank(B) + rank(C) = 2 + 2 = 4$. By first transforming the coupled matrix into the row-reduced echelon form and then determining the ranks of the major and minor matrices of the coupled matrix, respectively, we can also get the rank of A. rref(m) $= \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{8}{3} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ E $= \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{8}{3} \\ 0 & 0 & 0 \end{pmatrix}$ D $= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

Note that rank(E) = 2 and rank(D) = 2. As rank(A) = rank(D) + rank(E) = 2 + 2 = 4[MBI12]

Numerous aspects of rank of matrix can be extended to the rank of rhotrix, as can be seen from the definition of rank of a rhotrix. Specifically, we possess the following:

Theorem : 2.2. Assume that $Q_n = < b_{ij}, d_{kl} >$ and $R_n = < a_{ij}, c_{kl} >$.Let , represent any pair of $n-$dimensional rhotrices, with $n \in 2\mathbb{Z}^+ + 1$.[San08]

Then,

1) rank(R_n) $\leq$ n;

2) $rank(R_n + Q_n) \leq rank(R_n) + rank(Q_n)$;

3) $rank(R_n) + rank(Q_n) - n \leq rank(R_n o Q_n)$;

4) rank($R_n o Q_n$) $\leq min\{rank(R_n), rank(Q_n)\}$

Proof. Assume that there are two n-dimensional rhotrices, $R_n =< a_{ij}, c_{kl} >$ and $Q_n =< b_{ij}, d_{kl} >$, where n $\in 2\mathbb{Z}^+ + 1$.. Row-column multiplication of rhotrices is the multiplication taken into consideration for propositions (3) and (4). 1) According to the rhotrix's rank definition,

$$\text{rank}(R_n) = \text{rank}(a_{ij}) + \text{rank}(c_{kl}).$$

Since (a_{ij}) is a matrix of order ($\frac{n+1}{2}$) and (c_{kl}) is a matrix of order ($(\frac{n+1}{2})$ - 1) by applying the corresponding properties of rank of a matrix we get,

Therefore we get, $rank(R_n + Q_n) \leq rank(R_n) + rank(Q_n)$.

3) To prove the third statement, we apply corresponding inequalities of matrices, that is,

$$rank(AB) \geq rank(A) + rank(B) - n$$

where A is a $m \times n$ matrix and B is a $n \times p$ matrix. Now consider

$$rank(R_n o Q_n) = rank[(a_{ij})(b_{ij})] + rank[(c_{kl})(d_{kl}) \geq$$

$$[rank(a_{ij}) + rank(b_{ij}) - \tfrac{n+1}{2}] + [rank(c_{kl}) + rank(d_{kl}) - (\tfrac{n+1}{2}) + 1] =$$

$$rank(R_n) + rank(Q_n) - n.$$

4) For the last statement, consider

$$rank(R_n o Q_n) = rank[(a_{ij})(b_{ij}] + rank[(c_{kl})(d_{kl})]$$

$$\leq min(rank(a_{ij}), rank(b_{ij})) + min(rank(c_{kl}), rank(d_{kl}))$$

$$\leq min(rank(a_{ij}) + rank(c_{kl} + rank(d_{kl}))$$

$$= min(rank(R_n), rank(Q_n))[\text{San08}]$$

Chapter 3

NATURAL RHOTRICES

3.1 Natural Rhotrices

This section examines a collection of rhotrices with entries that are ordered natural numbers. The properties of this collection are analyzed and the findings are presented. Numerous researchers categorizes rhotrices into various types, including natural rhotrix set, real rhotrix set, complex rhotrix set, rational rhotrix set, and irrational rhotrix set.

Definition : 3.1. A rhotrix set with all of its entries being natural numbers is called a natural rhotrix set.

Definition : 3.2. If a rhotrix has an inverse, it is said to be invertible.

Many rhotrix $h(R) \neq 0$ are found to be invertible or non-singular rhotrix. A natural rhotrix, on the other hand, is unique. In other words, A^{-1} for which A is a natural rhotrix cannot be found.

Examples of Natural Rhotrix

Various representations of natural rhotrices are provided, based on their dimensions.

a)The following is a natural rhotrix of dimension five (R_5):

$$\mathbf{R_5} = \left\langle \begin{array}{ccccc} & & y & & \\ & r & s & t & \\ p & q & x & m & n \\ & g & k & l & \\ & & n & & \end{array} \right\rangle$$

where $x, y, r, s, t, p, q, g, n, m, k, l \in \mathbb{N}$

b)In general, the following represents a natural rhotrix of dimension n (R_n):

$$R_n = \left\langle \begin{array}{ccccccc} & & & a & & & \\ & & b & c & d & & \\ & .. & .. & .. & .. & .. & \\ .. & .. & .. & n^2+n+1 & .. & .. & .. \\ & .. & .. & .. & .. & .. & \\ & 2n^2+2n^{-2} & & 2n^2+2n+1 & & 2n^2+2n & \\ & & & 2n^2+2n+1 & & & \end{array} \right\rangle$$

where $a, b, c, \cdots, 2n^2 + 2n + 1$ for all $n' \in 2\mathbb{R} + 1$ and $n \in \mathbb{N}$ ($n' = 2n + 1$)

Lemma : 3.3. Let R_i be any natural rhotrix with dimension i. In this case, and the heart $h(R_i)$ is the middle value of the set of numbers that make the rhotrix $n = |R_i|, i = 1, 3, 5, \cdots$ [Ise16]

Proof. Since $n \in 2\mathbb{N} + 1$, then there exist middle value. so, if $n = |R_i|, 1, 3, 5, \cdots$ then for $i = 1$ is trivial. Then, $i = 3 \Rightarrow n = 5$ entries which are ordered natural numbers. Thus, the median is $3 = \frac{1}{2}(R_{2k+1} + 1)$. So, $i = 5 \Rightarrow n = 13$ entries which are ordered natural numbers. Thus, the median is $7 = \frac{1}{2}(|R_5|+1)$ So,$i = 2k+1$ $\Rightarrow$ n $= 2k^{2+2k+1}$ entries which are ordered natural numbers[Ise16]
Thus, the median is $n^2 + n + 1 = \frac{1}{2}\ (R_{2k+2} + 1)$.
The converse follows from the cardinality of R_n where $n \in 2\mathbb{N} + 1$.

Theorem : 3.1. Let R_n be any $n-$dimensional natural rhotrix. Then the following are equivalent:

a) The cardinality $|Rn|= \frac{1}{2}(n^2 + 1)$; $n \in 2\mathbb{N} + 1$

b) The last entry will be the value $2n\ '^2 + 2n\ '^1$ for all $n \in \mathbb{N}$.

c) The heart of $(h(R_n))$ is represented by

$h = \frac{1}{2}(|R_n| + 1)$, $n \in 2\mathbb{N} + 1$

d) $h(R_n)$ will be the value $n\ '^2 + 2n\ '^1 + 1$.[Ise16]

Proof. a) $\Rightarrow$ b) since $|Rn| = \frac{1}{2}(n^2 + 1)$ where $n \in 2\mathrm{N} + 1$, then for all $n\ ' \in \mathrm{N}$, $n = 2n\ ' + 1$.

Then $|R(n\,' + 1)| = 2n\,'^2 + 2n\,' + 1)$

b) $\Rightarrow$ c)

Since the last entry is $2n\,'^2 + 2n\,' + 1$ and is old then by lemma, the middle value is

$\frac{2n\,'^2+2n\,'+1}{2} + \frac{1}{2} = \frac{1}{2}(|R_n| + 1) \forall n \in 2\mathbb{N} + 1$

c) $\Rightarrow$ d)

Given that $h(R_n) = \frac{1}{2}(|R_n|+1)$ $\forall n \in 2\mathbb{N}+1$ and letting n = $2n\,'+1$ gives ,

$h(R_n) = n\,'^2 + n\,' + 1$

d) $\Rightarrow$ a)

Since $h(R_n) = n\,'^2 + n\,' + 1$ and by lemma , $h(R_n) = |R_n| + 1$, then $|R_n| = \frac{1}{2}(n^2 + 1)$[Ise16]

3.2 Determinant And Co determinant Function

Lemma : 3.4. Let $|A|$ be a determinant function of A, $|AB| = |A||B|$, and let A and B be any natural rhotrices of dimension n.

Proof. Let $|A| = h(A)$ and $|B| = h(B)$ then $|AB| = |h(A)h(B)|$ $= |A||B|$.

There are parallels between the idea of minor matrices and the co-determinant function. However, the co-determinant function could not always be the same in natural rhotrices of dimension 3 (R_3). A higher natural rhotrix must first be reduced to a chain of R_3 known as the minor rhotrices in order to determine its co-determinant function. Next, as was previously indicated, the determinant function of every minor rhotrix is assessed. The determinant functions are summed up in accordance with the division or reduction of these minor rhotrices along the major column or major row. Whether a natural rhotrix is summed up along the major row or the major column, the outcome stays the same for well-ordered entries.

Example : 3.1. Determine the natural rhotrices determinant and co-determinant below

$$\mathbf{R_5} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 2 & 3 & 4 & \\ 5 & 6 & 7 & 8 & 9 \\ & 10 & 11 & 12 & \\ & & 13 & & \end{array} \right\rangle$$

solution:

h(A) = 7

We then determine the co-determinant function. The first step along the major column :

$$Codet(A) = \left\langle \begin{array}{ccc} & 1 & \\ 2 & 3 & 4 \\ & 7 & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & 7 & \\ 10 & 11 & 12 \\ & 13 & \end{array} \right\rangle = 3 + 11 = 14$$

Now along minor row

$$\text{Codet(A)} = \left\langle \begin{array}{ccc} & 4 & \\ 5 & 6 & 7 \\ & 10 & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & 4 & \\ 7 & 8 & 9 \\ & 12 & \end{array} \right\rangle = 6 + 8 = 14$$

3.3 Index of Rhotrices

The index of natural rhotrix A is the number of minor rhotrices of dimension three that can be derived, along the major row, from A. This index is a whole number or better still a natural number. For example the index of R_3 is 1 and of R_5, R_7 and R_9 are 2,3 and 4, respectively. Appropriately, the index of R_1 is zero.[7]n[Ami10]

Theorem : 3.2. Given any rhotrix R the $Codet(R) = \rho h(R)$, where ρ where ρ is the index of R a natural number[Ami10]

Proof. Using mathematical induction, we prove the theorem. Given that n $\geq$ 3, n $\in 2\mathbb{N}$ +1, and R_n are the natural numbers from which the number of R_3 that can be calculated is represented by the index of a natural rhotrix. Codet(R_3) = h(R_3) is the result for n = 3, as the codet(R_3) is always the det(R_3) = h(R_3). For n = 3, the lemma holds. Two minors of R_3 exist for n = 5. Now, when n = 3, then, codet(R_3) = h(R_3) since the codet(R_3) is necessarily the det(R_3) = h(R_3). By Lemma Implies that $\rho = 1$, So, the equation is true for n = 3. For n = 5, then we have two minors of R_3. That is, codet(R_5) = 2h(R_5) implies that $\rho = 2$. So, the equation is true for n = 5. For n = 7, Next, we have three minors of R_3, then codet(R_7) = 3h(R_7) Thus, when n = 7, the equation is valid.and $\rho = 3$.Then, for n = 2k + 1, Then, it is true for n = 2k + 1 and $\rho =$ k. For n = 2k + 3, codet(R_{2k+3}) = codet($R_{2(k+1)+1}$) = k + 1h($R_{2(k+1)+1}$) Then, it is true for n = 2k + 3 and $\rho =$ k + 1. Hence, the equation is true for all value of n $\geq$ 3 and ρ a natural number.[Ami10]

Theorem : 3.3. Theorem Giving any natural rhotrix R codet(R) = $\frac{\rho}{2}(|R| + 1)$ where ρ is the index and $|R|$ is the cardinality of R, and $|Rn| = \frac{1}{2}(n^2 + 1)$[Ami10]

Proof. Since codet(R) = ρh(R) and by Lemma , determinant function is h(R). Then, the result follows from the theorem 3.[Ami10]

□

Chapter 4

CONCLUSION

Introduced as an extension of matrix theory, rhotrix theory is study related to linear algebra. We are still in the early stages of developing the theory of rhotrices. Next, we defined a few of the primary functions of Rhotrix. From there, we could conclude that because most rhotrix operations are similar to matrix operations and also we discovered that the set of all rhotrices forms a group structure under heart-oriented multiplication. Studying the rank of a rhotrix was made easier by converting it to a unique matrix known as a coupled matrix.

The last chapter we discussed the characteristics of the natural rhotrix set and introduced the ideas of co-determinant function, index of natural rhotrices. Ultimately, our effort led us to the conclusion that thinking theory and matrix theory are closely related.

The link between the coupled matrix and rhotrix allows for the application of several matrix theory principles to this rhotrix theory.Research on rhotrices will continue and yield further contributions to the fields of mathematics, science, and technology as a whole.

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