GAME THEORY

Dissertation submitted

in partial fulfilment of the requirements for the

MASTER'S DEGREE

in

MATHEMATICS

by

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MARCH 2024

BHARATA MATA COLLEGE , THRIKKAKARA

CERTIFICATE

This is to certify that the dissertation entitled, **GAME THEORY** is a bonafide record of the work done by Ms. **LIYA ROSE VARGHESE** under my guidance as partial fulfillment of the award of the degree of **Master of Science in Mathematics** at Bharata Mata College , Thrikkakara affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

Date: 06/07/2024
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DECLARATION

I hereby declare that the work presented in this project is based on the original work done by me under the guidance of Dr.Joby Mackolil, Assistant Professor, Department of Mathematics, Bharata Mata College, Thrikkakara and has not been included in any other project submitted previously for the award of any degree.

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ACKNOWLEDGEMENT

First of all, I would like to thank our Almighty God for giving us the determination to complete this project and to improve ourselves . The courage to make this work done with the strength, time, and effort that the proponents have.

Secondly, I want to convey my heartfelt appreciation and gratitude to our guide Dr.Joby Mackolil (Assistant Professor) for giving enough information and guiding through the project. In regard with this we would like to thank Dr.Lakshmi C (H.O.D of Mathematics) for the continuous support,patience and motivation.

I'm really grateful to all my friends and family for their great support and encouragement through out this period.

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ABSTRACT

Game theory is the systematic study of strategic decision-making in situations where the outcome depends on the actions of multiple individuals or parties. It provides a mathematical framework for analyzing and predicting the behavior of players in various contexts such as Economic markets (e.g., auctions, oligopoly) ,Nash equilibrium (stable states where no player can improve by unilaterally changing their strategy) ,Pareto optimality and Auctions .

Game theory has applications in various fields, including economics, political science, sociology, biology, computer science, and management, helping us better understand and predict human behavior in complex situations.

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Chapter 1

INTRODUCTION

The mathematical theory of games was first developed as a model for situations of conflict. It gained widespread recognition in the early 1940's when it was applied to the theoretical study of economics by the mathematician John von Neumann and the economist Oskar Morgenstern in their book Theory of Games and Economic Behavior. Since then its scope has been broadened to include co- operative interactions as well and it has been applied to the theoretical aspects of many of the social sciences.

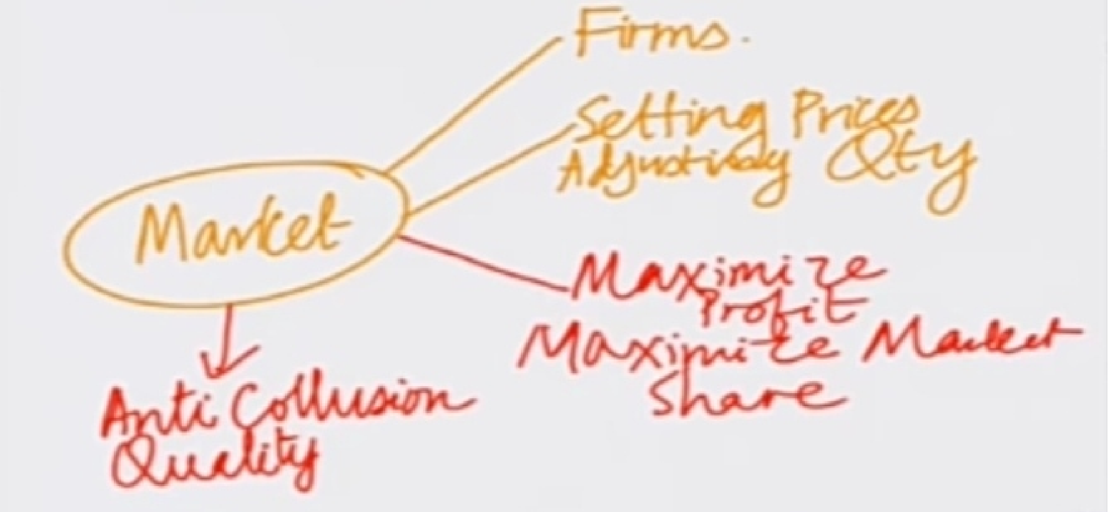
The goal with which game theory works is mainly with the action of the players involved and their respective outcomes or results. Also game theory has various applications in real life such as in economics to analyze market others include auctioning and the competition between companies.In biology to understand about the evolution and also about different populations and their behaviour . And mainly in this fast moving world game theory is related to computer sciences like the designing of the algorithms for the network route for cyber security and for artificial intelligence also.

Off which the application of game theory in economics is been seen in chapter three . Overall the field of game theory provides the mankind excruciating tools for the understanding and also to predict the strategic behavior in numerous areas. Thus making game theory an important area of study in various disciplines.

1.1 PRELIMINARIES

Firstly let's go through the basic definitions

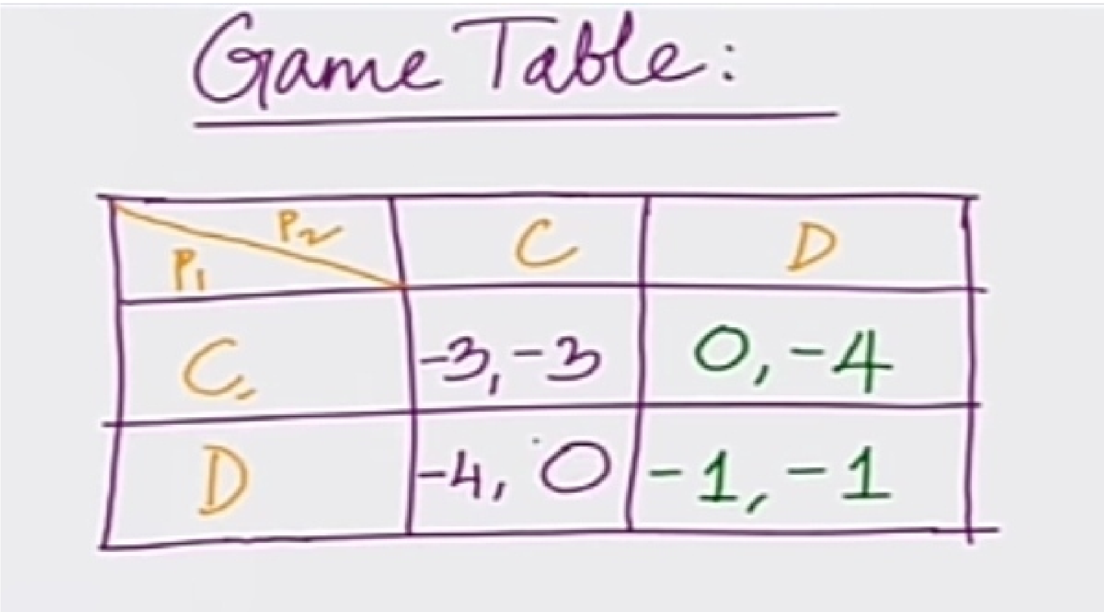
- Game:is a competitive activity in which multiple agents compete to maximize their profit according to a set of rules.
- Player:is any participant in a game who has non trivial set of strategies.
- Strategy:is an action or a plan for playing a game.
- Payoff:is the outcome of a game that depends upon the selected strategies of the player.
- Optimal Strategy: is basically the strategy that provides best payoff for the player in the game.
- A game could be modelled in many real life scenarios such as in politics in wireless communications in and in economic markets as well
- Now let's consider a case of the economic market in which a game is modelled:



1.2 Prisoner's Dilemma

- Two prisoners are accused of a major crime where there is no evidence or eyewitness account of them committing the crime.

- Both the prisoners are interrogated in separate rooms such that no communication is allowed between them.
- So the only possibility is one of them has to confess or both. The possible action of a prisoner is to confess(C)or to deny(D).
- If both deny then one year of sentence for both.
- If both confess then a sentence of 3 years for both.
- If one confess and other deny the one who confessed is allowed to walk free and the one who deny get a sentence of four years.
- Now let's observe the game table for this game of Prisoners Dilemma



Game Table:

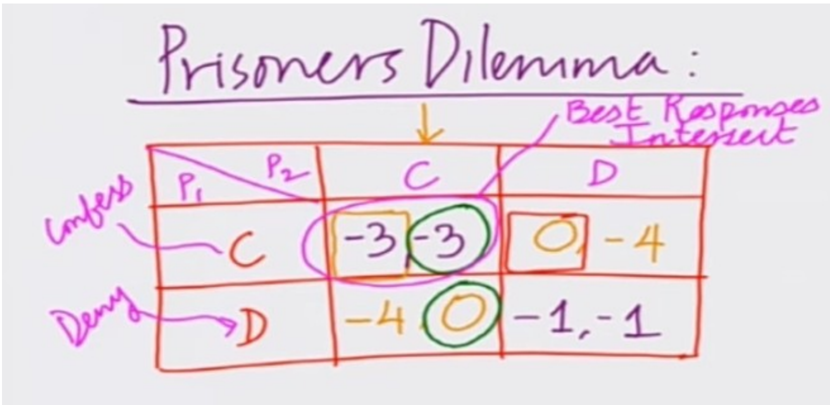
P_1 \ P_2	C	D
C	-3, -3	0, -4
D	-4, 0	-1, -1

- Prisoners Dilemma is a Strategic action that is the payoff is determined by the action of both the players.

1.3 Nash Equilibrium

- Best Response of a player i given the fixed action of all the other player, it is denoted as:

- BR_1(C) =C (BR of prisoner one, given the action of other prisoner to confess is to confess)
- BR_1(D) =C
- BR_2(C) =C
- BR_2(D) =C
- The above responses are known as best response dynamic.
- Now let's look at the nash equilibrium for Prisoner's dilemma:

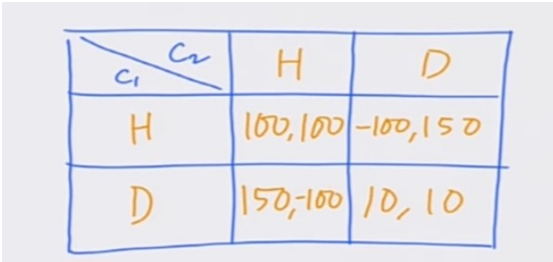


- Nash Equilibrium:is the intersection of best responses.
- In Nash equilibrium each player is playing his best response to the actions of all the other players.
- Other properties of NE are: In nash equilibrium outcome from which no player has an incentive to deviate unilaterally (or by himself) .
- Nash equilibrium is self enforcing.
- Nash equilibrium is a no regret outcome

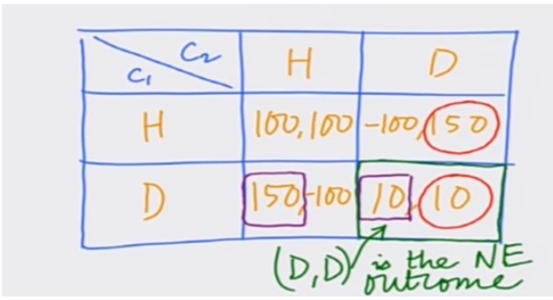
- Pareto Optimal:Given any outcome if there is no other outcome such that both players can simultaneously improve their payoffs is pareto optimal outcome.

1.4 Dominant Strategy

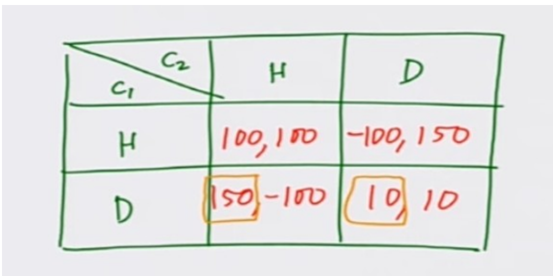
- Cold War Game :Consider two countries C_1 and C_2 involved in a cold war.C_1 and C_2 can choose from two different actions either to focus their resource on health of the citizens or to improve their defence.



- While analysing the best responses we came to see that(D, D) is the intersection of BR which is the Nash Equilibrium.

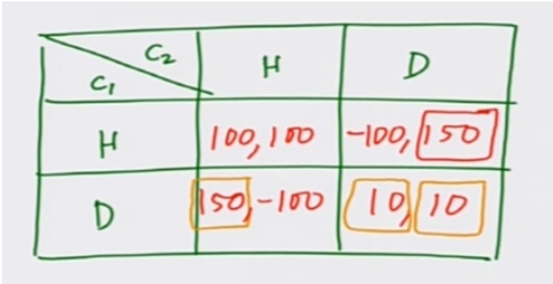


- Now let's see what is the dominant strategy:



- If C_2 invests in H then the BR of C_1 is to choose D
- If C_2 invests in D then the BR of C_1 is to choose D

- Irrespective of the action choose by C_2 the BR of C_1 is to choose D. Such a strategy which is always the BR irrespective of the action of other player is known as Dominant Strategy.
- D is a Dominant Strategy for C_1.

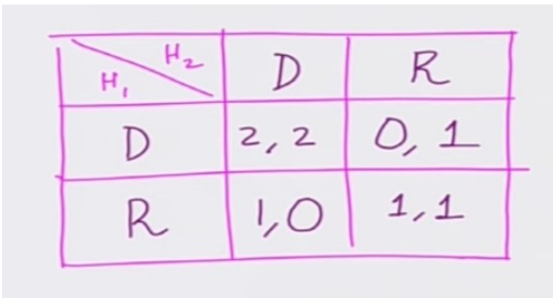


C_1 \ C_2	H	D
H	100, 100	-100, 150
D	150, -100	10, 10

- If C_1 invests in H then the BR of C_2 is to choose D.
- If C_1 invests in D then the BR of C_2 is to choose D.
- Therefore choosing D is the Dominant Strategy of C_2.

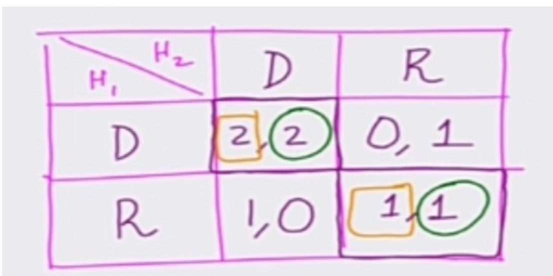
1.5 Coordination Game

- Hunting Game: Two hunters H_1 and H_2 can choose to hunt deer(D) or a rabbit(R) . Also as deer is larger when compared to rabbit hunting deer requires more coordination.



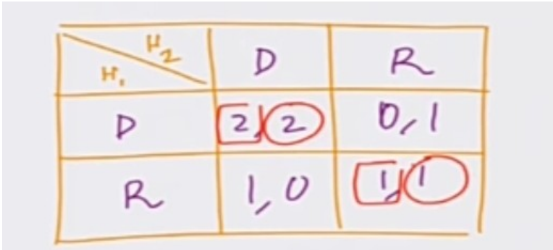
H_1 \ H_2	D	R
D	2, 2	0, 1
R	1, 0	1, 1

- while analysing the Best Responses:



H_1 \ H_2	D	R
D	2, 2	0, 1
R	1, 0	1, 1

- we can see that BR intersect at both (D, D) and (R, R) . Hence we have two Nash Equilibria in this case.
- Now let's analyse whether there is a dominant strategy or not:



H_1 \ H_2	D	R
D	2, 2	0, 1
R	1, 0	1, 1

- if H_2 chooses D the BR of H_1 is to choose D.
- if H_2 chooses R the BR of H_1 is to choose R.(hence the action of H_1 depends upon the action of H_2)
- if H_1 chooses D the BR of H_2 is to choose D.
- if H_1 chooses R the BR of H_2 is to choose R. (hence the action of H_2 depends upon the action of H_1)
- Hence no Dominant strategy for a Coordination Game.
- This is where the Coordination is important, the Best response of a player depends upon the action of other player. Hence the coordination between the two hunters is important to yield a better payoff of(D, D) than of (R, R) .

Chapter 2

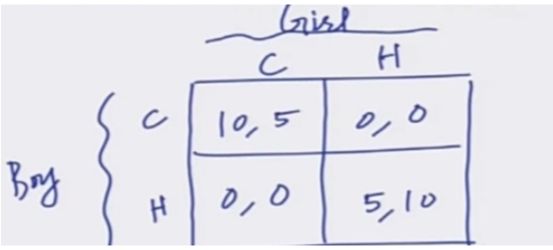
BAYESIAN GAME

2.1 Introduction

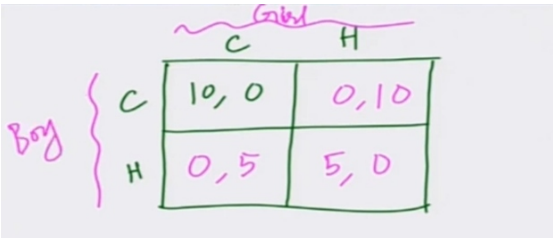
The main motivation behind Bayesian Games is that normally each player knows the exact payoff of all the other players as the game table is provided. In several games, the payoffs of other players are not known for example Auction in which each player knows his own payoff he might not know the payoff of the other player. So there is an uncertainty regarding the payoffs of the other player. These games in which there is an uncertainty regarding payoffs of the other player is known as Bayesian Games.

2.1.1 Example of Bayesian Game

- Battle of Sexes (BoS) :Is a game between a boy and a girl where the boy(P_1) and girl(P_2) either chooses to watch Cricket(C) or watch a movie Harry Potter(H) . Now consider the case when boy is uncertain about the mood or payoff of girl. That is the girl can either be interested (I) or uninterested (U) in watching cricket or movie with boy. So there are two types of girl player(P_2) which are (I) and (U) . We randomly assume that the probability of girl interested is half and the probability of girl uninterested is half .
- P(I) =1/2
- P(U) =1/2
- Girl of type1 is interested in watching C or H with the boy.



- Girl of type2 is uninterested in watching C or H with boy. So girl prefers to watch C or H alone while boy prefers to watch C or H together.



- So there is one type of boy and two types of girl, girl of type(I) and girl of type (U) . Hence there is different game table regarding each type.
- Battle of Sexes is Bayesian in nature since player1/boy is uncertain regarding the payoff of player2/girl. That is payoffs depends on the girl is of type(I) or that of type(U) .

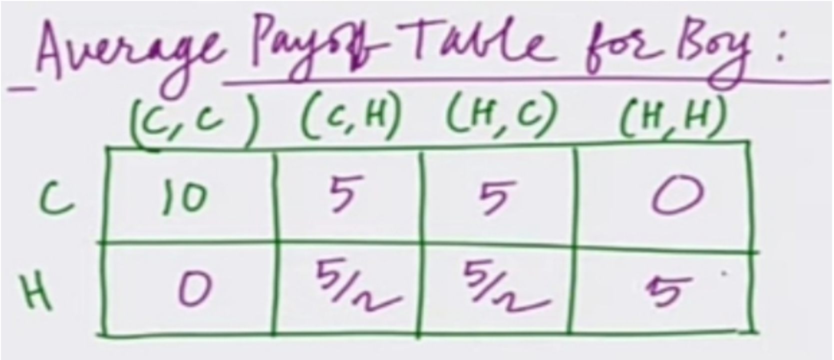
2.1.2 Application Bayesian Battle of Sexes

The most important thing in a Bayesian game is to assign a strategy to each player of each type.Remember in Bayesian battle of sexes the boy can either choose C or H, the girl of type(I) can choose C or H and girl of type (U) can choose C or H. So we have to assign a strategy to each player of each type.

- Let us consider boy choosing C.
- Let us consider girl of type(I) is choosing C.
- Also let girl of type (U) also chooses C.
- Strategy of girl can be represented as (C, C) where the first entry represent the strategy of girl of type(I) and the second entry

represent the strategy of girl of type(U) .

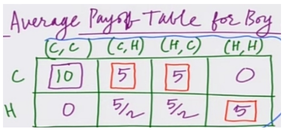
- **Now we will compute the payoff of the boy corresponding to the strategy of the girl. That is U_b(C, (C, C)) denotes the payoff of the boy when boy chooses C corresponding to the strategy (C, C) of the girl of type(I) and type(U) respectively.**
- **Even if the boy chooses C there is uncertainty regarding who he is playing with. Therefore we have to find the average payoff of boy with respect to the probabilities of different types of girl player.**
- **U_b(C, (C, C))=P(I)×U_b (C, C)+ P(U) ×U_b(C, C)**
- **U_b(C, (C, C))=1/2×10+1/2×10=10**
- **Now let's find U_b(H, (C, C)) denote the payoff of the boy when the boy chooses H corresponding to the strategy (C, C) of the girl of type(I) and type(U) respectively.**
- **U_b(H, (C, C))=P(I)×U_b (H, C)+ P(U) ×U_b(H, C)**
- **U_b(H, (C, C))=1/2×0+1/2×0=0**
- **Now let's find U_b(C, (C, H)) denote the payoff of the boy when the boy chooses C corresponding to the strategy (C, H) of the girl of type(I) and type(U) respectively.**
- **U_b(C, (C, H)) =P(I)×U_b (C, C)+P(U) ×U_b(C, H)**
- **U_b(C, (C, H))=1/2×10+1/2×0=5**
- **Now let's find U_b(H, (C, H)) denote the payoff of the boy when the boy chooses H corresponding to the strategy (C, H) of the girl of type(I) and type(U) respectively.**
- **U_b(H, (C, H))=P(I)×U_b (H, C)+P(U) ×U_b(H, H)**
- **U_b(H, (C, H))=1/2×0+1/2×5=5/2**
- **Similarly we can also consider other strategy choices of the girl (H, C) and(H, H) and compute average payoff of boy.**
- **The average payoff of boy**



Average Payoff Table for Boy :

	(C, C)	(C, H)	(H, C)	(H, H)
C	10	5	5	0
H	0	5/2	5/2	5

- Now let's analyse the Best responses of the boy : If the girl chooses (C, C) then the BR of boy is to choose C because it gives a payoff of 10. If the girl chooses (C, H) then the BR of boy is to choose C because it gives a payoff of 5. If the girl chooses (H, C) then the BR of boy is to choose C because it gives a payoff of 5. If the girl chooses (H, H) then the BR of boy is to choose H because it gives a payoff of 5.



Average Payoff Table for Boy

	(C, C)	(C, H)	(H, C)	(H, H)
C	10	5	5	0
H	0	5/2	5/2	5

Figure 2.1: Bayesian BR

- Now let's analyse the payoffs of the girl player: If the boy is choosing C then the BR of the girl of type(I) is to choose C. If the boy is choosing H then the BR of the girl of type(I) is to choose H. If the boy is choosing C then the BR of the girl of type(U) is to choose H. If the boy is choosing H then the BR of the girl of type(U) is to choose C.

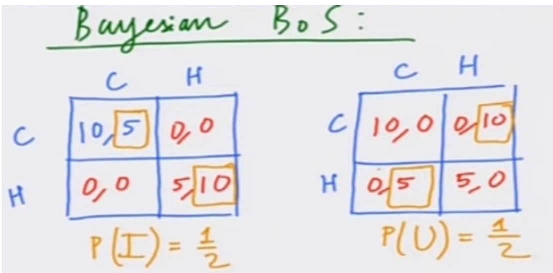


Figure 2.2: BR of girl

- Bayesian Nash Equilibrium(BNE) : From figure 1 of Bayesian BR,

let's check whether (C, (C, C)) is BNE ?(In BNE each player is playing their BR with each other) so C is the BR of the boy against (C, C) of the girl, is (C, C) the BR of the girl against the boy who chooses C. From figure 2 if the boy choose C then the BR of the girl of type (I) is to choose C, if the boy choose C then the BR of the girl of type (U) is to choose H, hence (C, (C, C)) is not BNE.

- **Now whether (C, (C, H)) is BNE? C is the BR of the boy against (C, H) of the girl, is (C, H) the BR of the girl against the boy who chooses C. From figure 2 if the boy choose C then the BR of the girl of type (I) is to choose C, if the boy choose C then the BR of the girl of type (U) is to choose H, hence (C, (C, H)) is BNE.**
- **Now whether (C, (H, C)) is BNE? C is the BR of the boy against (H,C) of the girl, is (H, C) the BR of the girl against the boy who chooses C. From figure 2 if the boy choose C then the BR of the girl of type (I) is to choose C not H , hence (C, (H, C)) is not BNE.**
- **Now whether (H, (H, H)) is BNE? H is the BR of the boy against (H, H) of the girl, is (H, H) the BR of the girl against the boy who chooses H. From figure 2 if the boy choose H then the BR of the girl of type (I) is to choose H, if the boy choose H then the BR of the girl of type (U) is to choose C not H, hence (H, (H, H)) is not BNE.**
- **Hence (C, (C, H)) is Bayesian Nash Equilibrium.**

2.2 Bayesian Cournot Game

Cournot game models market competition between two firms F_1 and F_2 .Now let's introduce uncertainty regarding the production cost of firm two. Firm 1 has a production cost of C per unit. However firm 2 is of two types. Firm 2 of type low has a production cost of 1/2C and probability of low is half. Firm 2 of type high has a production cost of C and probability of high is half.

- **Inverse Demand Curve is the price per unit p as the function of quantities q_1 and q_2 produced by firms 1 and 2 is given below.**

Inverse Demand Curve

Price p=(a-(q_1+q_2)) .

where q_1 is quantity produced by firm 1 and q_2 is quantity produced by firm 2.In a Cournot game if both the firms are producing strategic substitutes where one quantity can be more or less readily substituted for the other. Therefore the price per unit decreases with the total quantity q_1+q_2 produced by both the firms. So as q_1+q_2 increases price per unit decreases.

- **Payoff to each firm j is given as**
- **= price per unit×quantity-cost of production**
- **=(a-(q_1+q_2)) q_j-$c_j q_j$**

2.2.1 Bayesian Nash Equilibrium of Cournot Game

In a Bayesian Cournot game firm 1 produces quantity q_1 and firm 2 is of two types , type low(L) produces quantity q_2^L and type high(H) produces quantity q_2^H.

Payoff to Firm 2 of type L

=(a-(q_1+q_2^L)) q_2^L-(1/2)Cq_2^L

=aq_2^L-$q_1 q_2^L$-$(q_2^L)^2$-(1/2)Cq_2^L

We have to differentiate this expression and set it into zero to find the best response of firm2 of type(L) .

- **Differentiate wrt q_2^L and set zero**

a-q_1-2q_2^L-(1/2)C=0

$(q_2^L)^*$=(a-(1/2)C-q_1)/2

Payoff to firm 2 of Type H is:

(a-(q_1+q_2^H)) q_2^H-Cq_2^H

=aq_2^H-$q_1 q_2^H$-$(q_2^H)^2$-Cq_2^H

- **differentiate wrt q_2^H and set zero to find the best response of firm2 of type (H).**

a-q_1-2q_2^H-C=0

$(q_2^H)^*$=(a-q_1-C) /2

- **Now we have to find the payoff of firm1, for firm1 there is uncertainty regarding the payoff of firm2. So for firm1 we have to compute the average payoff.**
- **Payoff of firm1, corresponding to type low of firm2 is (a-(q_1+q_2^L)) q_1-Cq_1**
- **Payoff of firm1, corresponding to type high of firm2 is (a-(q_1+q_2^H)) q_1-Cq_1**
- **Average payoff of firm1is1/2(a-(q_1+q_2^L)) q_1-Cq_1+1/2(a-(q_1+q_2^H)) q_1-Cq_1 Differentiate wrt q_1 and set to zero to find the BR q_1:**
- **1/2(a-2q_1-q_2^L-C)+1/2(a-2q_1-q_2^H-C)=0**
- **by solving the above we get the BR of firm 1**

2q_1^*=1/2(a-C-q_2^L) +1/2(a-C-q_2^H)

q_1^*=$\frac{a-c}{2}$-$\frac{1}{4}$(q_2^L+q_2^H)

- **Also we have**

$(q_2^L)^*$=**(a-(1/2)C-q_1)/2**

$(q_2^H)^*$=**(a-q_1-C) /2**

- **By substituting these values in q_1^* we get**

q_1^*=$\frac{a-c}{2}$-$\frac{1}{4}$($(q_2^L)^*$+$(q_2^H)^*$)

=$\frac{a-c}{2}$-$\frac{1}{4}$(**(a-(1/2)C-q_1)/2+(a-q_1-C) /2)**

- **by solving the above we get BR quantity of firm 1**

q_1^*=**(a-(5C/4))/ 3**

- **substitute the value of q_1^* in $(q_2^L)^*$ we get the BR of quantity of type low $(q_2^L)^*$**

$(q_2^L)^*$=**(a-(1/2)C-q_1^*) /(2)**

=$\frac{(a-(1/2)C-1/3(a-(5C/4)))}{2}$

=$\frac{a}{3}$-$\frac{c}{24}$

- **Similarly substitute the value of q_1^* in $(q_2^H)^*$ we get the BR of quantity of type high $(q_2^H)^*$**

$(q_2^H)^*$=**(a-C-q_1^*) /(2)**

=$\frac{(a-C-1/3(a-(5C/4)))}{2}$

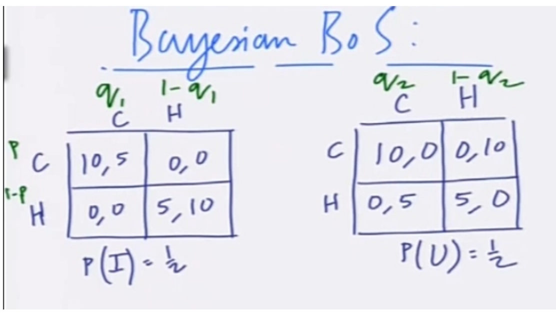
=$\frac{a}{3}$-$\frac{7c}{24}$

- **Hence the Bayesian Nash Equilibrium for Cournot game is**

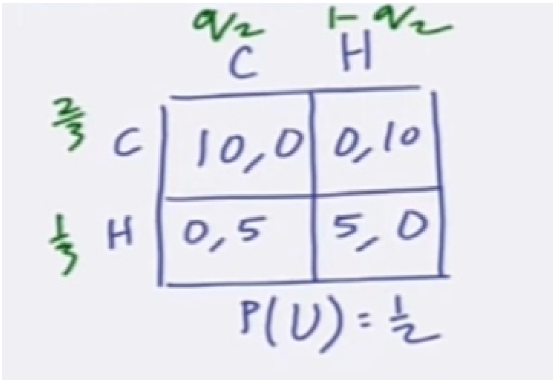
((a-(5c/4))/ 3,(a/3-c/24, a/3-7c/24))

2.3 Mixed Strategy Bayesian Games

Players of different types can use mixed strategy instead of pure strategy. Let's look at Bayesian Battle of Sexes game a game between a boy and a girl where the boy(P_1) and girl(P_2) either chooses to watch Cricket(C) or watch a movie Harry Potter(H) . Now consider the case when boy is uncertain about the mood or payoff of girl. That is the girl can either be interested (I) or uninterested (U) in watching cricket or movie with boy. So there are two types of girl player(P_2) which are (I) and (U) . We randomly assume that the probability of half the girl is interested and probability of half the girl is uninterested. The boy is choosing C and H with probabilities p and 1-p respectively and girl of type (I) is choosing C and H with probabilities q_1 and 1-q_1 respectively also girl of type (U) is choosing C and H with probabilities q_2 and 1-q_2 respectively.



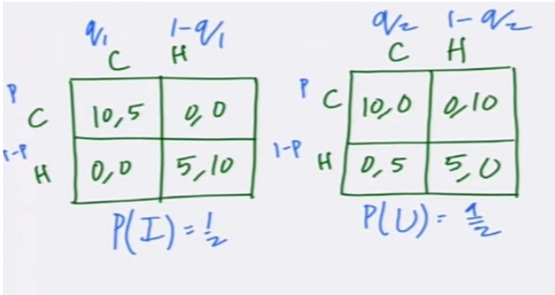
Payoff of girl of type (I) to choose C is with probability p is 5 Payoff of girl of type (I) to choose C is with probability 1-p is 0.So payoff of girl of type (I) to always choose C is 5p+0(1-p) =5p. Similarly payoff of girl of type (I) to always choose H is 0p+10(1-p) =10(1-p) . Girl of type(I) will choose a mixed strategy or randomly mix C and H if these two payoffs are equal. That is 5p= 10(1-p) , 15p=10 , p=2/3. Therefore the mixed strategy employed by the boy is 2/3, 1/3.That is mixing C, H with probabilities 2/3, 1/3 respectively.



Payoff of girl of type(U) for choosing C is 0.2/3+5.1/3=5/3.Similarly Payoff of girl of type(U) for choosing H is 10.2/3+0.1/3=20/3. For girl of type (U) choosing H yields a strictly greater payoff than choosing C. So girl of type(U) is always choosing H with probability 1.Which implies q_2 =0 and 1-q_2=1

- Now what is remaining is to find the mixed strategy employed by girl if type(I) that is to find q_1 and 1-q_1.We have to look at the payoffs of boy for choosing C and H, and the payoffs has to be equal as he is choosing a mixed strategy. Payoffs to the boy for choosing C when he meets an interested girl with probability half is 1/2(10.q_1+0.1-q_1) =5.q_1, when he meets the girl of type(U) she is always choosing H, so his payoff corresponding to that is 0.1/2.So the payoffs to the boy for choosing C is 5.q_1+0.1/2=5.q_1.
- The payoffs of boy when he chooses H with probability half when he meets girl of type(I) is 1/2(0.q_1+5.1-q_1) , with probability half when he meets girl of type (U) girl of type(U) is always choosing H, his payoff for choosing H is 1/2.5.The payoffs of boy when he chooses H is 5/2.(1-q_1) + 5/2 .
- As the boy is choosing the mixed strategy the payoff to choose C is equal to payoff to choose H . So 5.q_1=5/2.(1-q_1) + 5/2 , 10q_1=5(1-q_1) +5, 15q_1=19, q_1=2/3 so 1-q_1 = 1-2/3=1/3.
- Mixed strategy for girl of type (I) is (2/3,1/3) .
- The Bayesian Mixed Nash Equilibrium is ((2/3, 1/3),((2/3, 1/3),(0, 1))) .

- We can derive the Bayesian Mixed Nash equilibrium by another way by the assumption girl of type(U) is mixing.



The girl of type(U) is choosing a mixed strategy her payoff to always choosing C is 0.p+5(1-p)=5(1-p) . Her payoff to always choosing H is 10.p+0.(1-p) =10.p . As she's her payoffs from both C and H are equal 10.p=5(1-p), 15p=5, p=1/3 and 1-p=2/3. Therefore the mixed strategy employed by the boy is (1/3, 2/3) .

- If p=1/3 and 1-p=2/3 the payoff of girl of type(I) for always choosing C is 5.1/3+0.2/3=5/3 and her payoff for always choosing H is 0.1/3+10.2/3=20/3 . Her payoff to H is strictly greater than that of C which means the probability with which she is choosing H is 1 and the probability with which she is choosing C is 0.Therefore girl of type(I) always choosing H implies q_1= 0 and 1-q_1=1.

- Now it remains to find the mixture of girl of type(U) q_2 and 1-q_2 and this can be find by looking at the payoffs of the boy. The payoff of the boy corresponding to C with probability half when meeting girl of type (I) is always choosing H, therefore bis payoff is 1/2.0 and with probability half he is meeting girl of type(U) his payoff is 1/2.(10.q_2+0.1-q_2) . So the net payoff of boy corresponding to C is 1/2.0+1/2.(10.q_2+0.1-q_2) =5q_2

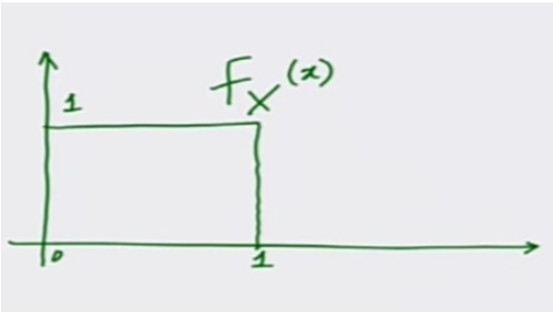
- The payoff to boy for always choosing H, with probability half of meeting girl of type(I) girl of type(I) is always choosing H so his payoff is 1/2.5 and with probability half he is meeting girl of type(U) whose probability is q_2 and 1-q_2 corresponding payoff of the boy is 1/2(0.q_2+5.1-q_2) . So the net payoff of boy corresponding to H is 5/2+1/2(5.1-q_2) =5/2+5/2(1-q_2).

- Since he is mixing both the payoffs must be equal.5q_2=5/2+5/2(1-q_2), 10q_2 =5+5(1-q_2) implies q_2=2/3 and 1-q_2=1/3.Therefore mixture of girl of type(U) is (2/3, 1/3).

- Another mixed strategy for Bayesian Nash Equilibrium of Battle of Sexes Game ((1/3,2/3), ((0, 1) , (2/3, 1/3))).

2.4 Auctions Modelled As Bayesian Games

That is games with uncertainty. We are going to consider random variable(RV) which is uniformly distributed in [0,1].And the probability density function of the uniform random variable in[0, 1] is by f_X(x) is 1 if $0 \leq x < 1$ and is 0 otherwise.



Further this RV takes a value in the interval[a, b] is given by $\int_a^b f_X$(x) dx.

- let's consider an example what is the probability that the uniform RV takes a value between [1/4,1/2] is $\int_{1/4}^{1/2} f_X$(x) dx $=\int_{1/4}^{1/2}$ 1 dx =1/2-1/4=1/4. Therefore the probability that the uniform RV takes a value between [1/4,1/2] is 1/4.

- what is the probability that the uniform RV takes a value between [0,1/2] is $\int_0^{1/2} f_X$(x) dx $=\int_0^{1/2}$ 1 dx =1/2.

- To generalize this consider any interval [a,b] which lies in [0, 1] the probability that the uniform RV takes a value between [a,b] is $\int_a^b$ f_X(x) dx $=\int_a^b$ 1 dx =b-a.

- Therefore probability that the uniform RV lies in any interval [a, b] fully contained in [0,1] is b-a, the length of the interval. Therefore the probability that it takes a value in [0, 1]=1-0=1.
- So this RV can take values uniformly in the interval [0, 1]as the RV is uniformly distributed in [0,1].

2.4.1 Nash Equilibrium of Sealed Bid First Prize Auction

Consider a two player auction P_1 and P_2 in which these players submit their individual bids b_1 and b_2 respectively for the object being auctioned. These bids are sealed in envelopes. Therefore each player does not know the bid of the other player. So P_1 does not know b_2 of player2 and P_2 does not know b_1 of player1. Player with the highest bid wins the auction and he pays the amount equal to his bid to get the object this is known as First Price Auction. If $b_1 \geq b_2$ then P_1 wins the auction and pays his bid b_1 to get the object, player2 who has lost the auction does not pay anything. On the other hand if b_2¿b_1 then P_2wins the auction and pays his bid b_2 to get the object, player1 who has lost the auction does not pay anything. In addition each player has a private valuation for the objects this might not be same as that of bid. Let v_1 and v_2 be the valuations of player1 and player2 respectively for the objects to be auctioned. These valuations are private implies P_1 does not know valuation v_2 and P_2does not know valuation v_1 . Hence valuations are private. However some statistical information are know these private valuations are uniformly distributed in the interval [0, 1]. That is probability density of v_1 and v_2 are uniformly distributed in the interval [0, 1].

- Auction is Bayesian as there is uncertainty regarding the valuation of the other player.We have to find the Nash equilibrium bidding strategy of each player. We will demonstrate that the Nash equilibrium bidding strategy is b_1=1/2v_1 and b_2=1/2v_2 . Therefore each player bidding half his valuation is the Nash equilibrium bidding strategy.We will demonstrate b_1=1/2v_1 and b_2=1/2v_2 are the BR against each other.

- **Let's assume that player2 is bidding b_2=1/2v_2.Then what is the BR bid b of player1. π(b) denotes payoff of player1 as a function of bid b.If player1 wins the auction that is $b \geq b_2$ if he wins with bid b then he pays an amount equal to bid b therefore he looses his bid b however he gains with respect to his valuation, then the net payoff of player1 is v_1-b.On the other hand if player1 looses the auction $b \leq b_2$ the net payoff is zero because he doesn't pay any thing neither he gets the object. Therefore the average payoff of player1 is given as probability (winning) $\times$(v_1-b) +probability (losing)$\times$0=Pr(Win) $\times$(v_1-b).**

- **Payoff of player1 as a function of bid b is π(b)=Pr(Win) $\times$(v_1-b). Now what is Pr(Win) , the probability of winning the auction for the player1 is $b \geq b_2$=1/2v_2 , $b \geq$1/2v_2 implies $v_2 \leq$2b. Since v_2 is distributed uniformly in [0, 1] we have v_2 in [0,2b].Therefore if Player1 should win $v_2 \leq$2b , v_2 lies between 0 to 2b. Probability of v_2 lies in [0, 2b] is $\int_0^{2b} Fv_2(v_2)\, dv_2 = \int_0^{2b} 1\, dv_2$ = [2b-0]=2b**

- **Therefore Pr(Win)=2b**

- **Therefore Pr(Win)=2b**

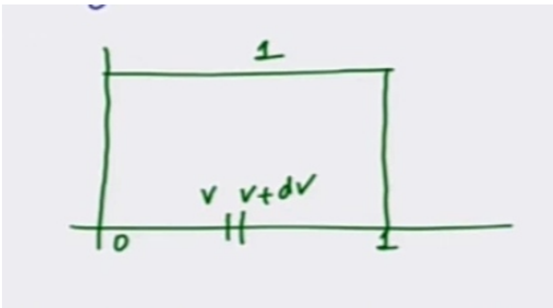
- **Therefore the net payoff of player 1 as the function of bid b is π(b)=Pr(Win) $\times$(v_1-b)=2b$\times$(v_1-b)=2bv_1-2b^2.Now we have to find the b for which the payoff is maximum for that we have to differentiate this with b and set it into zero.So 2v_1-4b=0 , b=1/2v_1.Implies the BR bid b^*=1/2v_1. If b_2=1/2v_2 then the BR bid of player1 is b_1=1/2v_1.**

- **Using the similar procedure it can be showed that if player1 is bidding b_1=1/2v_1 then b_2=1/2v_2 is the BR of player2.**

- **Therefore b_1=1/2v_1 and b_2=1/2v_2 are best responses to each other, hence the Nash Equilibrium of the sealed bid First Price Auction.**

2.5 Expected Revenue of First Prize Auction

The revenue the First Prize auction is expected to bring to the auctioneer or the average price the object which is being auctioned fetches . The Nash equilibrium for First prize auction is b_1=1/2v_1 and b_2=1/2v_2 . Since the player with highest bid wins the auction, pays a amount equal to his bid. Hence the revenue is the maximum of the bids b_1 , b_2 . revenue=max[b_1 , b_2] and from NE substitute the values, revenue=max [b_1=1/2v_1,b_2=1/2v_2] , revenue= 1/2max[v_1,v_2]. So revenue to the auctioneer is half of the maximum of the valuations. Since these valuations v_1,v_2 are random variable which are distributed uniformly in the interval [0, 1], we have to find half of the average value of v_1,v_2. Let v_1 and v_2 be independent valuations distributed uniformly in the interval [0, 1]. Consider a infinitesimal small interval between v, v+dv



- **What is the probability that the max[v_1,v_2] lies in the infinitesimal small interval [v, v+dv] , this can happen in two possible ways. Scenario 1: v_1 is the maximum and lies in [v, v+dv] and v_2 lies in [0, v] . So the probability of this event equals Pr=Pr(v_1€[v, v+dv]) ×Pr(v_2€[0, v]) . Since v_1 and v_2 be independent valuations distributed uniformly in the interval [0, 1] the Pr(v_1€[v, v+dv]) is length of the interval dv , similarly the Pr(v_2€[0, v]) is the length of the interval v. Pr = dv×v , so the probability that v_1 is the maximum and lies in [v, v+dv] and v_2 lies in [0, v] is v.dv .**

- **Scenario 2:v_2 is the maximum and lies in [v, v+dv] and v_1 lies in [0, v] .Now the probability that this event happens is Pr=Pr(v_1€[0, v]) ×Pr(v_2€[v,v+dv]). The Pr(v_1€[0, v]) is v and the Pr(v_2€[v,v+dv]) is dv . Pr=v×dv .**

- **So the probability that the max[v_1,v_2] lies in the infinitesimal small interval [v, v+dv] is vdv+vdv =2vdv.**

- **Average revenue corresponding to max[v_1,v_2] lies in the interval [v, v+dv] is 1/2.v×2vdv = v^2dv**

- **So the net average revenue to the auctioneer is $\int_0^1 v^2$ dv =1/3-0=1/3 .**

- **Therefore for a Sealed bid First prize auction between two players with valuations v_1,v_2 whit are distributed uniformly in the interval [0, 1] the expected average revenue to the auctioneer at the Nash equilibrium is 1/3.**

2.6 Bayesian Second Price Auction

Consider an auction with two players P_1 and P_2. Let b_1 , b_2 be their respective bids. If $b_1 \geq b_2$ then P_1 wins the auction and pays the second highest bid b_2. On the other hand if b_2¿b_1 then P_2 wins the auction and pays the second highest bid b_1.Each player P_1 and P_2 has a private valuation v_1 and v_2 respectively. v_1 and v_2 are independent and uniformly distributed in the interval [0, 1].

2.6.1 Nash Equilibrium of Second Price Auction

Now we will demonstrate that b_1=v_1 and b_2=v_2 is the Nash equilibrium of Second Price auction. That is each player is bidding his true valuation is the Nash equilibrium in Second Price auction.

- **To demonstrate this NE let's start with the assumption b_2=v_2 that is player2 is bidding his true valuation. Let's now find the best response bid b of player1. For that consider two cases. Case 1:$v_1 \geq v_2$, let's try to find out the BR bid of player1, remember player2 is bidding b_2=v_2. If $b \geq v_2$ then player1 wins and pays the second highest bid b_2=v_2. Therefore the net payoff of player1 is valuation minus the amount he is paying implies v_1-v_2 which is greater than or equal to zero because $v_1 \geq v_2$. On the other hand if player1 bids $b \leq b_2$=v_2 then player1 looses the auction and his net payoff is zero** as P_1 doesn't pay anything or get anything.Therefore we see that any bid $b \geq v_2$ is the best response. In particular b=v_1 is a best response.

- Case 2: Let's now consider $v_1 \leq v_2$,and player2 p_2 is bidding b_2=v_2.If player1 p_1 bids $b \geq v_2$=b_2 then p_1 wins the auction and he pays the second highest bid b_2=v_2. Therefore the net payoff of player1 is valuation minus the amount he is paying implies v_1-v_2 which is less than or equal to zero because $v_1 \leq v_2$. On the other hand if player1 bids any bid $b < v_2$=b_2 then he looses the auction and payoff of player1 is zero. So if player1 bids any bid $b \geq v_2$=b_2 then he gets a negative payoff, if P_1 bids any bid $b < v_2$=b_2 then he gets a payoff zero. Therefore P_1 bids any bid $b < v_2$=b_2 is the best response. In particular b=v_1 is a best response.

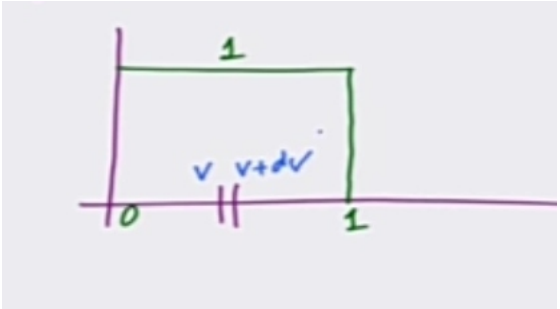
- Therefore player P_2 is bidding b_2=v_2 then b_1=v_1 is a best response of player P_1.

- Similarly it can be shown that if player P_1 is bidding b_1=v_1 then b_2=v_2 is a best response of player P_2.

- So b_1=v_1 and b_2=v_2 are the best response of each other hence b_1=v_1 and b_2=v_2 are Nash equilibrium of the Second Price auction. That is each player is bidding his true valuation is the Nash equilibrium in Second Price auction.

2.6.2 Expected Revenue of Second Price Auction

We know that b_1=v_1 and b_2=v_2 are Nash equilibrium of the Second Price auction. If $b_1 \geq b_2$ implies $v_1 \geq v_2$ then P_1 wins the auction and pays the second highest bid b_2=v_2. On the other hand if b_2¿b_1 implies v_2¿$v_1$then P_2 wins the auction and pays the second highest bid b_1=v_1.Therefore the revenue to the auctioneer in this Bayesian Second Price auction is min [b_1 , b_2]= min[v_1,v_2] .These valuations v_1,v_2 are random variable which are distributed uniformly in the interval [0, 1].



- What is the probability that min[v_1,v_2] lies in the interval [v, v+dv].We have two for this:
- Case 1: v_1 $\leq$$v_2$ that is v_1 lies in the interval [v, v+dv] and v_2 lies in interval [v+dv, 1] .The probability of this event is Pr=Pr(v_1€[v, v+dv])×Pr(v_2€[v+dv, 1])= dv× (1-v-dv) =dv.(1-v-dv) as dv is infinitesimal small value we can neglect it so we get Pr=dv(1-v)=(1-v)dv.
- Case 2: When v_2 <v_1 that is v_2 lies in the interval [v, v+dv] and v_1 lies in interval [v+dv, 1] . Therefore the probability of this event equals Pr=Pr(v_2€[v, v+dv])×Pr(v_1€[v+dv, 1])= dv× (1-v-dv) =dv.(1-v-dv) as dv is infinitesimal small value we can neglect it so we get Pr=dv(1-v)=(1-v)dv. The net probability thatmin[v_1,v_2] lies in the interval [v, v+dv] is (1-v)dv+(1-v)dv=2(1-v) dv.
- The revenue of the auctioneer is min[v_1,v_2].Since minimum lies in [v, v+dv] revenue of the auctioneer is v. Therefore the expected revenue =probability that min[v_1,v_2] lies in the interval [v, v+dv] times v = 2(1-v)dv×v =2(1-v)vdv.Now we have to integrate $\int_0^1$2(v-1)v dv =$\int_0^1$2(v-v^2) dv=2[1/2-1/3]=2.1/6=1/3.
- The expected revenue of the auctioneer in Bayesian Second Price auction is 1/3.
- Note: For Bayesian First Price auction and Bayesian Second Price auction when we observe the revenue of the auctioneer it is both same that is 1/3. This is termed as Revenue Equivalence Principle, irrespective of the auction format the revenue of the auctioneer is same.

Chapter 3

APPLICATION OF GAME THEORY IN ECONOMICS : COURNOT DUOPOLY

INTRODUCTION : Here we analyse the market game between two different companies or two different firms which produces two different products in a market. This is a very popular and well established game known as $CournotDuopoly$.

3.1 Cournot Duopoly

The name duopoly indicates competition between two firms by producing two different goods which have same utility or the same applicability. In 1838 $COURNOT$ is the economist who introduced this market game

3.1.1 Strategic Substitutes

Market based strategic interaction:consider two firms which are producing two goods which are related closely where the consumer can substitute one good for the other which indicate that the consumers don't have a strong preference for any of the two.

For Example: Consider the two different soft drinks which are available in the market which are related closely or two different clothing line such that the consumers do not have a strong preference over one cloth

versus the other. Such items or goods which can be easily or readily substituted for one another is called *StrategicSubstitutes.*

- **Strategic substitutes are basically two goods or two items which are related closely with respect to their functionality and applicability such that the consumers don't have a strong preference for one another. These strategic substitutes are often used to model the fast moving consumer goods where people do not have a strong preference over one another.**
- **Now in Cournot duopoly we are going to consider the competition that happens between two firms in a market which are producing goods which are strategic substitutes.**

3.1.2 Market Game

Consider the two firms F_1 and F_2 producing quantities S_1 and S_2 of these strategic substitutes, where S_1 and S_2 are the strategies or actions. The strategy of each firm is to determine the quantity of goods produced. So S_1 is the action of firm 1 and S_2 is the action of firm 2.So as the competition takes place between two firms this is a Duopoly where the two firms compete by producing two different quantities of goods where firm 1 is producing S_1 of the good and firm 2 is producing S_2 of the good where these two good are strategic substitutes With the aim to increase the profit and capture more of the market.

- **The utility function of each firm can be modelled as below:**

  The price function , that is the price per unit is given as

  $p(S_1,S_2) = \mathbf{A}\text{-}\mathbf{B}(S_1 + S_2)$

  where A and B are positive constants which depends upon the market and particular good. Also the price is decreasing with the quantity , which is coined by the term inverse demand function.

That is the price depends on $S_1 + S_2$ .So as S_1 increases or S_2 increases the price decreases. Because these two goods are strategic substitutes one can be easily replaced by the other which is seen in the inverse demand price function.

- **The cost per unit be given by C.**
- **The total cost of firm 1 F_1 is CS_1**
- **The total cost of firm 2 F_2 is CS_2**
- **Now the payoff of each firm can be obtained as follows:**

$U_1(S_1,S_2)$ =**Total Revenue - Total Cost**

=**(price per unit**$\times$ **quantity) - Total Cost**

=**(A-B(**$S_1 + S_2$**))**$\times$ S_1 **- C**S_1

= **(A-C-B(**$S_1 + S_2$**))** S_1

- **The payoff or total profit of firm 1 is**

$U_1(S_1,S_2)$=S_1 **(A-C-B(**$S_1 + S_2$**))**

- **Similarly the payoff of firm 2 is**

$U_2(S_2,S_1)$=S_2 **(A-C-B(**$S_1 + S_2$**))**

We can observe that this is a strategic interaction between two firms F_1 and F_2. For instance the profit of F_1 depends not only on the quantity S_1 produced by firm 1 but also on the quantity S_2 produced by firm 2 and similarly the profit of F_2 depends not only on the quantity S_2 produced by firm 2 but also on the quantity S_1 produced by firm 1 .

3.1.3 Analysis of Cournot duopoly

Now let's see the behavior of these two firms or about the quantities which are produced by the two firms by producing the good which is a strategic substitute and have same utility.

- **Let's consider infinite set of quantities which are continuous. So the set of quantities S_1 and S_2 can be real numbers belonging to the continuous interval. So we cannot draw a game table to find the best response and there by the Nash equilibrium. So we use differential calculus to find the best response and there by the Nash equilibrium.**

- **To find the best response S_1^* of firm 1 , differentiate $U_1(S_1,S_2)$ with respect to S_1 and set to zero:**

$U_1(S_1,S_2)=S_1$ $(\mathbf{A}\text{-}\mathbf{C}\text{-}\mathbf{B}(S_1 + S_2)\ )$

$U_1(S_1,S_2)$ $=\mathbf{A}S_1\text{-}\mathbf{C}S_1\text{-}\mathbf{B}S_1^2\text{-}\mathbf{B}S_1S_2$

$(\partial \mathbf{U}_1/\partial \mathbf{S}_1)$ $=\mathbf{A}\text{-}\mathbf{C}\text{-}2\mathbf{B}S_1\text{-}\mathbf{B}S_2$

$\mathbf{A}\text{-}\mathbf{C}\text{-}2\mathbf{B}S_1\text{-}\mathbf{B}S_2=0$

$S_1^*=(\mathbf{A}\text{-}\mathbf{C}\text{-}\mathbf{B}S_2)/2\mathbf{B}$

- $S_1^*=(\mathbf{A}\text{-}\mathbf{C}\text{-}\mathbf{B}S_2)/2\mathbf{B}=\mathbf{B}R_1(S_2)$ **. Hence we got the best response quantity of firm 1 to be produced in response to the quantity S_2 produced by firm 2.**

- **Again by symmetry we get the best response quantity of firm 2 to be produced in response to the quantity S_1 produced by firm 1 as**

$S_2^*=(\mathbf{A}\text{-}\mathbf{C}\text{-}\mathbf{B}S_1)/2\mathbf{B}=\mathbf{B}R_2(S_1)$

- **As we the best response we can easily find the nash equilibrium, as nash equilibrium is the intersection of best responses. That is:**

S_1^*=**B**$R_1(S_2^*)$

S_2^*=**B**$R_2(S_1^*)$

- **Now we can write the system of equations to solve the nash equilibrium:**

S_1^***=(A-C-B**S_2^***)/2B —-(1)**

S_2^***=(A-C-B**S_1^***)/2B —-(2)**

- **Now substitutes (2) in (1)**

S_1^***=(A-C)/2B - 1/2(**S_2^***)**

S_1^***=(A-C)/2B - 1/2((A-C)/2B - 1/2**S_1^***)**

(3/4)S_1^***=(A-C) / 4B**

S_1^***= (A-C) / 3B**

- **By symmetry**

S_2^***= (A-C) / 3B**

- **So nash equilibrium quantities are**

S_1^***= (A-C) / 3B**

S_2^***= (A-C) / 3B**

So NE of Cournot duopoly =((A-C) / 3B,(A-C) / 3B)

3.2 Graphical analysis of Cournot duopoly

- **Let A=10, B=C=1**

  S_1^***=(A-C-B**S_2**)/2B**

  S_1^***=4.5-(1/2)**S_2

  Similarly S_2^***=4.5-(1/2)**S_1

- **On the X-axis we have** S_1^* **and on Y-axis we have** S_1^* **.**

- S_1^***=4.5-(1/2)**S_2 **if** S_2**=9 then we have** S_1^***= 0 , also if** S_2**=0 then we have** S_1^***= 4.5. Hence the red line represent** S_1^***= B**$R_1(S_2)$ **.**

- **Similarly** S_2^***=4.5-(1/2)**S_1 **if** S_1**=9 then we have** S_2^***= 0 , also if** S_1**=0 then we have** S_2^***= 4.5. Hence the yellow line represent** S_2^***= B**$R_2(S_1)$ **.**

- S_1^* **and** S_2^* **are the best response quantities of firm 1 and firm 2. Therefore their intersection gives the nash equilibrium. Which is** S_1^***=**S_2^***= (A-C) / 3B = (10-1)/3 = 3.Therefore NE=(3,3) .**

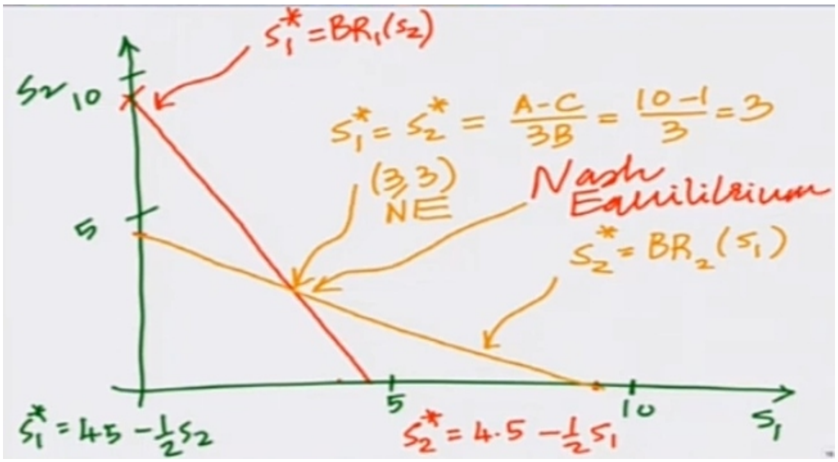


Figure 3.1: NE of Cournot duopoly

3.3 Further Analysis of graph of Cournot duopoly

3.3.1 Dominated Strategy

- **We know that S_1^*=4.5-(1/2)S_2**
- **Also if S_2=0 then we have the BR of firm 1 is S_1^*= 4.5 . So for all possible values of S_2 the best response S_1^* lies between 0 and 4.5. So these quantities which are greater than 4.5 is never an best response , that is S_1 greater than 4.5 is never an best response of firm 1. So quantities greater than 4.5 is never used by the firm 1 which means they are the dominated strategy.**
- **Also we know that S_2^*=4.5-(1/2)S_1**
- **If also if S_1=0 then we have the BR of firm 2 is S_2^*= 4.5 . So for all possible values of S_1 the best response S_2^* lies between 0 and 4.5. So these quantities which are greater than 4.5 is never an best response , that is S_2 greater than 4.5 is never an best response of firm 2. So quantities greater than 4.5 is never used by the firm 2 which means they are the dominated strategy.**

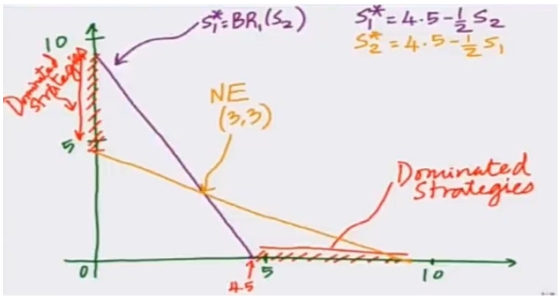
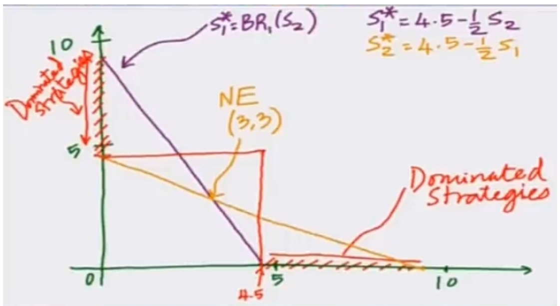


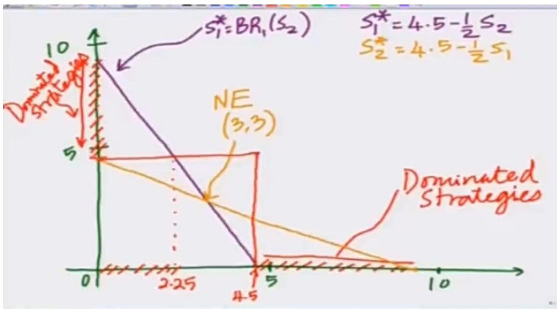
Figure 3.2: Dominated Strategy

- **So the best response of firm 1 always lies between 0 and 4.5 and the best response of firm 2also always lies between 0 and 4.5 . So the game of Cournot duopoly can be reduced to the box given below as the best response S_1^* and S_2^* of firm 1 and firm 2 respectively lies between 0 and 4.5 .**

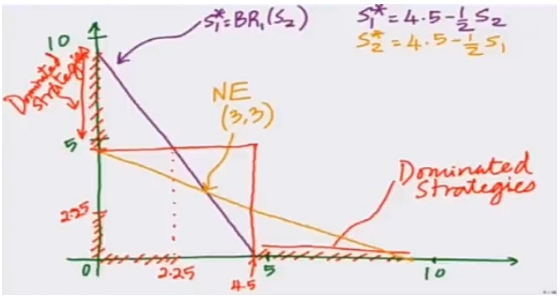


- Hence this game of Cournot duopoly can be restricted to the smaller box in which S_1 is restricted to 0 to 4.5 and similarly S_2 is restricted to 0 to 4.5 . So we can eliminate the dominated strategy and hence this game is reduced.

- We can observe that S_2 lies between 0 and 4.5 . Where S_1^*=4.5-(1/2)S_2 , when S_2=0 then S_1^*=4.5 also when S_2=4.5 then S_1^*=2.25 . After the removal of the dominated strategy S_2 lies between 0 and 4.5 , we can observe that the best response S_1^* lies between 2.25 and 4.5 . Hence the strategies S_1^* which lies between 0 and 2.25 becomes the dominated strategy and this could be eliminated as these strategies are no longer the best response.



- Similarly, we can observe that S_1 lies between 0 and 4.5 . Where S_2^*=4.5-(1/2)S_1 , when S_1=0 then S_2^*=4.5 also when S_1=4.5 then S_2^*=2.25 . After the removal of the dominated strategy S_1 lies between 0 and 4.5 , we can observe that the best response S_2^* lies between 2.25 and 4.5 . Hence the strategies S_2^* which lies between 0 and 2.25 becomes the dominated strategy and this could be eliminated as these strategies are no longer the best response.



- **Eliminating these dominated strategy from the reduced game we a smaller box colored orange .**

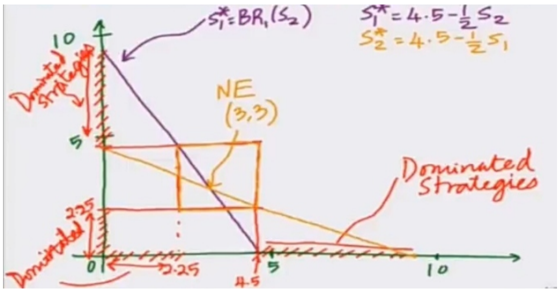


Figure 3.3: Reduced Game

- **Now again from this Reduced game we can eliminate the dominated strategy . So by repeating this process box will reduce furthermore.**
- **The box thus converges to the Nash equilibrium.**

3.4 Nash Payoff

- **We know S_1^*=S_2^*=(A-C) / 3B.**
- **For A=10, B=C=1 then S_1^*=S_2^*=3.**
- **Let's look at the nash payoff :**

$U_1(S_1^*,S_2^*)$ =S_1^***(A-C-B**(S_1+S_2))

$U_1(S_1^*,S_2^*)$ **=3(10-1-1(6)) = 9**

- **Therefore the Nash payoff for Cournot duopoly game is**

$U_1(S_1^*,S_2^*)$=$U_2(S_2^*,S_1^*)$=**9**

- **Now let's check is there any outcome which gives an higher payoff for both the players. To examine this let's consider the case when both firms are collaborating**

$U_1(S_1,S_2)$ =S_1(**A-C-B**(S_1+S_2)) **and**

$U_2(S_2,S_1)$ =S_2(**A-C-B**(S_1+S_2))

Therefore $U_1(S_1,S_2)$+$U_2(S_2,S_1)$=(S_1+S_2)(**A-C-B**(S_1+S_2)))

As the above quantity depends upon the sum S_1+S_2, **let** S_t=S_1+S_2

then$U_1(S_1,S_2)$+$U_2(S_2,S_1)$=S_t(**A-C-B**(S_t))

Hence the total utility as a function of S_t **can be written as**

$U_t(S_t)$=S_t(**A-C-B**(S_t))

$U_t(S_t)$=(**A-C**)S_t - **B**S_t^2

- **Now differentiating with respect to** S_t **to find the total quantity** S_t **in which sum utility** U_t **is maximum.**
- **differentiating with respect to** S_t **and equate to zero**

$\partial \mathbf{U}_t(S_t)/\partial \mathbf{S}_t$=(**A-C**)-**2B**$S_t$ = **0**

S_t^*=(**A-C**)/**2B**

- **Now let's assume that both the firms produces half of these quantities**

therefore S_1=S_2=(A-C)/4B

- **Consider the previous example when A=10, B=C=1 then**

  (A-C)/4B=9/4=2.25 . So both the firm can produce quantities S_1=S_2= 2.25 to get the high sum utility.

- **Now let's look the individual utility**

  U_1(9/4,9/4) =9/4(10-1-1(9/2))=9/4×9/2=81/8

- **U_1(9/4,9/4) = 81/8**

  U_1(9/4,9/4) = 81/9 × 9/8

  U_1(9/4,9/4) = 9 ×(a quantity greater than 1)

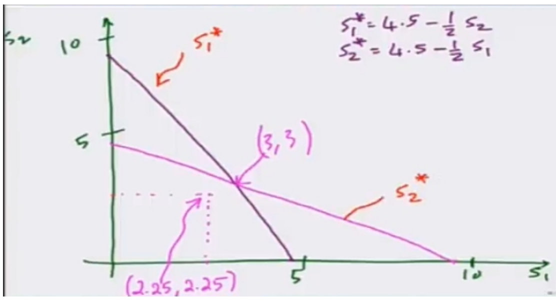
  U_1(9/4,9/4) = a quantity greater than 9 , which is greater than the nash payoff of 9 . So when we look the payoff at (9/4,9/4)each of them is getting a payoff of 81/8 which is greater than the nash payoff 9.

- **Hence the outcome S_1=S_2=9/4=2.25 yields the payoff which is higher for both.**

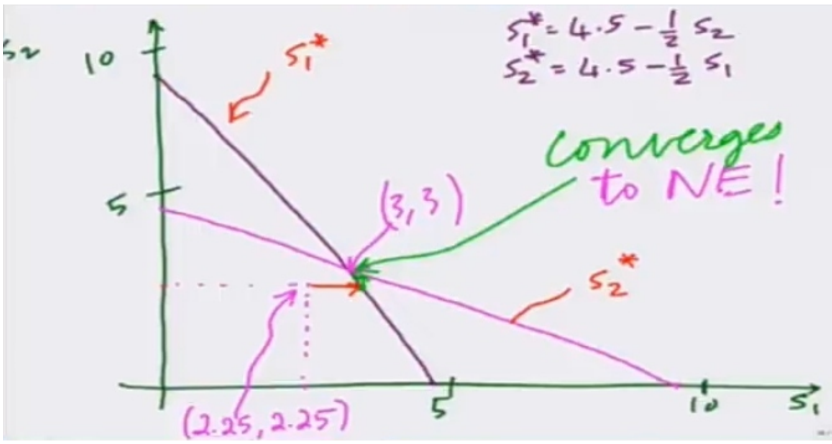
- **So both the of firms can increase their payoff strictly so the nash equilibrium is not pareto optimal.**

- **Why can't these two firms agree to make this quantity of 2.25 to yield a higher payoff. Because they cannot collude to produce the quantity 2.25 artificially to inflate the market price. As it is illegal by following the anti collusion law.**

- Now if both the firms implicitly agreed to produce this quantity of 2.25 then we have



- The moment when S_2 agree to produce a quantity of 2.25 . S_1 will deviate to it's best response the point shown by red arrow. Also the moment S_1 deviate to it's best response S_2 will also deviate to it's best response the point shown by green arrow. And in turn S_1 will return to it's best response and reaches the Nash equilibrium (3,3) . Therefore (2.25,2.25) is not self sustaining outcome as each firm will update it's strategy by playing the best response with the other firm and reaches or converges to NE as nash equilibrium is the only self sustaining outcome where is firm is competing it's best response against the other.



Chapter 4

CONCLUSION

Game theory offers valuable insights into decision-making in various fields, from economics to biology and beyond . Through the analysis of strategic interactions, game theory helps us understand how individuals or entities behave and make choices in competitive situations. However, game theory also has its limitations, such as assumptions of rationality and perfect information, which may not always hold in real-world scenarios. Nonetheless, the continued refinement and application of game theory contribute to our understanding of complex systems and provide practical tools for decision-makers in diverse contexts.

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Available at: https://youtube.com/playlist?list=PLDIJ2nwWsVZ1si=hs9In$_6 5B$