BETA AND GAMMA FUNCTIONS

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DECLARATION

We hereby declare that this project report entitled "BETA AND GAMMA FUNCTIONS" is a bonafide record of work done by us under the supervision of Anntreesa Josy and the work has not previously formed the basis for the award of any academic qualification fellowship or other similar title of any other university or board.

Place: Thrikkakara

Date: 23.04.2024

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CERTIFICATE

This is to certify that dissertation entitled "**BETA AND GAMMA FUNCTIONS"** submitted jointly by Mr. Varun A, Miss. Suja Jayakumar, Miss. Aiswarya V. S. in partial fulfilment of the requirements for the BSc Degree in Mathematics is a bonafide record of the studies undertaken by them under the supervision of the Department of Mathematics, Bharata Mata College, Thrikkakara during 2021-2024. This dissertation has not been submitted for any other degree elsewhere.

Place: Thrikkakara

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(Supervisor)

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CONTENTS

INTRODUCTION

The beta and gamma functions are two important mathematical functions that arise in various branches of mathematics, including calculus, analysis, and mathematical statistics. They are extensions of the factorial function and play crucial roles in solving problems involving integration, differentiation, and series representations. This project deals with the important properties and applications of beta and gamma functions along with the concept of Improper integrals and how the Beta and Gamma functions are closely connected to it and with each other.

First chapter being the introduction, the second one deals with Improper Integrals. Improper integrals are definite integrals where one or both the limits of integration extend to infinity or where the function being integrated has a singularity within the interval of integration. These integrals do not fit the standard definition of a definite integral, but they can still be evaluated by considering limits. The chapter also deals with the two types of improper integrals, their convergence and divergence and the convergence tests of two types of improper integrals.

The second chapter is about Beta functions. The Beta function, denoted by $\beta(x, y)$, is a function in mathematics that is used to extend the factorial function to real and complex numbers. We deal with basic definition and the convergence of beta function along with the elementary properties.

The third chapter is about Gamma functions. The gamma function, denoted by $\Gamma(z)$, is an extension of the factorial function to complex and real numbers (except negative integers). It is defined for all complex numbers except non-positive integers. In this chapter we deal with the basic definition, convergence of gamma function, the recurrence formula for gamma function and the relation between beta and gamma functions along with a few examples.

CHAPTER 1

IMPROPER INTEGRALS

Improper integrals are definite integrals in which one or both the limits of integration extend to infinity or where the integrand has a singularity within the interval of integration. These integrals do not fit the standard definition of a definite integral, but they can still be evaluated by considering limits.

There are two types of improper integrals:

1. Type 1: Infinite limits of integration.

Type 1 improper integrals have one or both limits of integration extending to infinity. For example:

$$\int_a^\infty f(x)dx$$

$$\int_{-\infty}^b f(x)dx$$

To evaluate the integrals, we take the limits:

$$\log_{c\to\infty} \int_a^\infty f(x)dx$$

$$\log_{c\to-\infty} \int_c^b f(x)dx$$

Example1:

$$\int_1^\infty \frac{1}{x^2}dx$$

Here, the function $\frac{1}{x^2}$ is unbounded as x approaches infinity. To evaluate this integral, we take the limit:

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^2}$$

Then, we can integrate the function from 1 to b, and take the limit as b approaches infinity. In this is case, the integral converges to 1.

2. Type 2: Discontinuities or singularities within the interval of integration.

Type 2 improper integrals arise when the function being integrated is undefined or unbounded within the interval of integration. Mathematically, a Type 2 improper integral is defined as:

$$\int_a^b f(x)dx$$

Where either the function $f(x)$ is unbounded or undefined at some point within the interval $[a, b]$. To evaluate a Type 2 improper integral, we identify the points of discontinuity or where the function becomes unbounded within the interval, and then we split the integral at those points. Then we compute the limit as those points approach each other.

Example:

$$\int_0^1 \frac{1}{\sqrt{x}}dx$$

Here, the function $\frac{1}{\sqrt{x}}$ is undefined at $x = 0$ within the interval [0,1]. To evaluate this integral, we rewrite it as a limit:

$$\lim_{a \to 0+} \int_a^1 \frac{1}{\sqrt{x}} dx$$

Then, we integrate the function from a small positive value a to 1, and take the limit as a approaches 0 from the right side. In this case, the integral converges to 2.

CONVERGENCE OR DIVERGENCE OF TYPE I IMPROPER INTEGRALS

Let $f(x)$ be an integrable and bounded function in a finite interval given by $a \leq x \leq$b. Then define

$$\int_a^\infty f(x)dx = \lim_{n \to \infty} \int_a^b f(x)dx.$$

The integral on the left is convergent or divergent based on whether the limit on the right exists or not.

$\text{Similarly, we define}$

$$\int_{-\infty}^b f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx$$

The integral is convergent or divergent based on whether the limit on the right exists or not.

So, we have

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

where c is any real number.

Here, $\int_{-\infty}^{\infty} f(x)dx$ is convergent if both the integrals on the right side $\int_{-\infty}^{c} f(x)dx$ and $\int_{c}^{\infty} f(x)dx$ exists and, $\int_{-\infty}^{\infty} f(x)dx$ is divergent otherwise.

EXAMPLE 1:

Test for the convergence of $\int_{1}^{\infty} \frac{1}{3x^2} dx$.

$$\int_{1}^{\infty} \frac{1}{3x^2} dx = \lim_{b\to\infty} \int_{1}^{b} \frac{1}{3x^2} dx = \lim_{b\to\infty} \left[-\frac{1}{3x}\right]_{1}^{b} = \frac{1}{3}\left[-\frac{1}{b} - (1)\right]$$

$$= \frac{1}{3} \lim_{b\to\infty} \left(1 - \frac{1}{b}\right) = \frac{1}{3}.$$

Therefore, $\int_{1}^{\infty} \frac{1}{3x^2} dx$ converges.

CONVERGENCE OR DIVERGENCE OF

TYPE II IMPROPER INTEGRALS

If the function $f(x)$ is unbounded only at the end point $x = a$ of the interval $a \leq x \leq b$, then we define

$$\int_{a}^{b} f(x)dx = \lim_{m\to 0+} \int_{a+m}^{b} f(x)dx.$$

$\int_{a}^{b} f(x)dx$ is convergent if the limit on the right exists and is divergent otherwise

If the function $f(x)$ is unbounded only at $x = b$ of the interval $a \leq x \leq b$, then we can define

$$\int_a^b f(x)dx = \lim_{m\to 0+} \int_a^{b-m} f(x)dx.$$

The integral $\int_a^b f(x)dx$ is convergent if the limit on right exists and is divergent otherwise.

If the function $f(x)$ is unbounded at an arbitrary point say c of the interval $a \le x \le b$, then we can define

$$\int_a^b f(x)dx = \lim_{m1\to 0+} \int_a^{c-m1} f(x)dx + \lim_{m2\to 0+} \int_{c+m2}^{b} f(x)dx$$

The integral on the right converges or diverges based on whether the limit on the right exists or not respectively.

Similar process is carried out if the function is unbounded at more than one point in the given interval.

EXAMPLE: Test the convergence, $\int_0^1 \frac{1}{x^2} dx.$

The function in this case is unbounded at $x = 0$. Therefore,

$$\int_0^1 \frac{1}{x^2} dx = \lim_{m\to 0+} \int_m^1 \frac{1}{x^2} dx = \lim_{m\to 0+} \left[-\frac{1}{x}\right]_m^1$$

$$= \lim_{m\to 0} \left(\frac{1}{m} - 1\right) = \infty$$

Therefore, the function is divergent.

CONVERGENCE TESTS FOR

TYPE I IMPROPER INTEGRALS.

Convergence tests for improper integrals of the first kind, which have infinite limits of integration, are essential in determining whether such integrals converge or diverge. Here are some common convergence tests used specifically for improper integrals of the first kind:

1. Comparison Test:

If $0 \leq f(x) \leq g(x)$ for $x \geq a$ (or $x \leq a$ for an integer with $a = -\infty$), where $g(x)$ is an integrable function, then:

- if $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges.
- If $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ diverges.

Example:

Test for convergence $\int_0^1 \frac{sinx}{\sqrt{x}}$dx

=> We know that $\left|\frac{sinx}{\sqrt{x}}\right| \leq \frac{1}{\sqrt{x}}$

since $\int_0^1 \frac{1}{\sqrt{x}}\, dx$ is convergent.

$\int_0^1 \frac{sinx}{\sqrt{x}} dx$ is convergent. [by comparison test]

2. Quotient test:

Suppose $f(x) \geq 0$ and $g(x) > 0$ for all $x \geq a$ (or $x \leq a$ for an integral with $a = -\infty$) and let

$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$ then,

- If L is a non-zero finite number i.e., if $L \neq 0$ or ∞, then $\int_a^\infty g(x)dx$ converge or diverge together.
- If $L = 0$ and $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges.
- If $L = \infty$ and $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges.

CONVERGENCE TESTS FOR

TYPE II IMPROPER INTEGRALS.

Considering the case where the function $f(x)$ is unbounded only at $x = a$ in the interval $a \leq x \leq b$. Similar tests are available where $f(x)$ is unbounded at $x = b$ and $x = x_0$, in the interval $a < x_0 < b$.

1. Comparison Test.

Let f and g be two positive functions which are unbounded at $x = a$ and such that $f(x) \leq g(x)$ where $a \leq x \leq b$, then,

- $\int_a^b f(x)dx$ converges, if $\int_a^b g(x)dx$ converges.
- $\int_a^b g(x)dx$ diverges, if $\int_a^b f(x)dx$ converges.

2. Quotient Test.

If $f(x) \geq 0$ and $g(x) \geq 0$ for $a \leq x \leq b$ if $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = L$, then

- If $L \neq 0$ or ∞, that is, if l is a non-zero finite number, then $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge or diverge together.
- If $L = 0$ and $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ converges.

- If $L = \infty$ and $\int_a^b g(x)dx$ diverges, then $\int_a^b f(x)dx$ diverges.

Example:

$$\int_{\mathbf{0}}^{\mathbf{1}} \frac{\mathbf{1}}{\boldsymbol{x}^{\mathbf{3}}(\mathbf{1}+\boldsymbol{x}^{\mathbf{2}})}\text{dx}$$

We know that $f(x) = \frac{\mathbf{1}}{\boldsymbol{x}^{\mathbf{3}}(\mathbf{1}+\boldsymbol{x}^{\mathbf{2}})}$ is not bounded at $x = 0$

Therefore, consider the function which is divergent $g(x) = \frac{1}{\boldsymbol{x}^{\mathbf{3}}}$

then $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\boldsymbol{x}^{\mathbf{3}}}{\boldsymbol{x}^{\mathbf{3}}(\mathbf{1}+\boldsymbol{x}^{\mathbf{2}})} = \lim_{x\to 0} \frac{1}{(1+x^2)} = 1$

we know that $\int_{\mathbf{0}}^{\mathbf{1}} \frac{\mathbf{1}}{\boldsymbol{x}^{\mathbf{3}}}$ dx is divergent.

Hence $\int_{\mathbf{0}}^{\mathbf{1}} \frac{\mathbf{1}}{\boldsymbol{x}^{\mathbf{3}}(\mathbf{1}+\boldsymbol{x}^{\mathbf{2}})}$dx is also divergent.

CHAPTER 2

BETA FUNCTION

The beta function, represented by the notation β(m,n), is a unique function in mathematics that is defined for positive real integers m and n. It is widely utilized in probability theory, integration theory, and many other areas of physics, especially quantum and statistical mechanics. The definition of the beta function is:

$$\beta(m,n) = \int_0^1 y^{m-1}(1-y)^{n-1}\,dy$$

Additionally, it is connected to the gamma function by:

$$\beta(m, n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

CONVERGENCE OF BETA FUNCTION

If $m \geq 1$ and $n \geq 1, \beta\ (m,n)$ is a proper integral.

If $0 < m < 1$ or $0 < n < 1, \beta\ (m,n)$ is an improper integral of second kind.

So $\int_0^1 y^{m-1}(1-y)^{n-1}\,dy \ = \ \int_0^{\frac{1}{2}} y^{m-1}(1-y)^{n-1}\,dy + \ \int_{\frac{1}{2}}^1 y^{m-1}(1-y)^{n-1}\,dy$

$$= A + B$$

where

$A \ = \ \int_0^{\frac{1}{2}} y^{m-1}(1-y)^{n-1}\,dy$ and

$B \ = \int_{\frac{1}{2}}^1 y^{m-1}(1-y)^{n-1}\,dy$

Convergence of A

Let $f(y) \ = \frac{(1-y)^{n-1}}{y^{1-m}}$ and

let $g(y) \ = \frac{1}{y^{1-m}}$

Then, $\frac{f(y)}{g(y)} = (1-y)^{n-1}$

Taking limit on both sides,

$$\lim_{y \to 0} \frac{f(y)}{g(y)} = \lim_{y \to 0} (1-y)^{n-1} = 1$$

Since $\int_0^{\frac{1}{2}} \frac{1}{y^{1-m}} dy$ converges,

So by quotient test,

$\int_0^{\frac{1}{2}} y^{m-1}(1-y)^{n-1}\, dy$ converges.

Convergence of B

Let $f(y) = \frac{y^{m-1}}{(1-y)^{1-n}}$ and

let $g(y) = \frac{1}{(1-y)^{1-n}}$

Then, $\frac{f(y)}{g(y)} = y^{m-1}$

Taking limit on both sides,

$$\lim_{y \to 1} \frac{f(y)}{g(y)} = \lim_{y \to 1} y^{m-1} = 1$$

Since $\int_{\frac{1}{2}}^{1} \frac{1}{(1-y)^{1-n}} dy$ converges

So by quotient test,

$\int_{\frac{1}{2}}^{1} y^{m-1}(1-y)^{n-1}\, dy$ converges.

Therefore,

$\int_0^1 y^{m-1}(1-y)^{n-1}\, dy$ converges.

KEY PROPERTIES OF BETA FUNCTIONS

1. **Symmetric property**

β(m, n) = β(n, m)

We know that,

$\beta(m,n) = \int_0^1 y^{m-1}(1-y)^{n-1}\,dy$ ----(1)

Put $1-y=x$

Then, $-dy = dx$

$$y:\ 0 \longrightarrow 1$$

$$x:\ 1 \longrightarrow 0$$

$$\therefore\ \beta(m,\ n) = \int_1^0 (1-x)^{m-1}\,x^{n-1}(-dx)$$

$$= -\int_1^0 (1-x)^{m-1}\,x^{n-1}dx$$

$$\left[\because\ -\int_a^b f(x)\ dx = \int_b^a f(x)\ dx\right]$$

Then, $\beta(m,\ n) = \int_0^1 (1-x)^{m-1}\,x^{n-1}dx$

$$= \int_0^1 (1-y)^{m-1}\,y^{n-1}dy$$

$$= \beta(n,m)$$

$\therefore \beta(m,n) = \beta(n,m)$

2. $\boldsymbol{\beta(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\,\theta cos^{2n-1}\theta\ d\theta}$

   we know that,

$\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx$ -----------(1)

Put, $x\ = sin^2\theta,\quad 1-x\ = cos^2\theta$

d$x\ =\ 2sin\theta cos\theta\ d\theta$

x: $0\rightarrow 1$

θ:$0\rightarrow \frac{\pi}{2}$

Equation (1) becomes,

$$\beta(m,n) = \int_0^{\frac{\pi}{2}} sin^2\theta^{m-1}cos^2\theta^{n-1}\,2sin\theta cos\theta\ d\theta$$

$$= 2 \int_0^{\frac{\pi}{2}} sin^{2m-1}\,\theta cos^{2n-1}\theta\, d\theta$$

3. **If m and n are positive integers,**

Then, β (m, n) = $\frac{(\boldsymbol{m-1})!(\boldsymbol{n-1})!}{(\boldsymbol{m+n-1})!}$

$$\beta\ (m,n)\ =\ \int_0^1 x^{m-1}\,(1-x)^{n-1}dx$$

$$=\ [(1-x)^{n-1}\frac{x^m}{m}]_0^1 + \int_0^1 (n-1)\,(1-x)^{n-2}\frac{x^m}{m}dx$$

$$= \frac{n-1}{m}\int_0^1 x^m\,(1-x)^{n-2}dx$$

$$= \frac{n-1}{m}\beta(m+1,n-1)$$

$$= \frac{n-1}{m}\cdot\frac{n-2}{(m+1)}\beta(m+2,n-2)$$

$$= \frac{n-1}{m}\cdot\frac{n-2}{m+1}\cdot\frac{n-3}{m+2}\cdots\frac{1}{m+n-2}\beta(m+n-1,1) \qquad ---(a)$$

Also, $\beta\ (m+n-1,1) =\ \int_0^1 x^{m+n-2}(1-x)^0\,dx$

$$= \int_0^1 x^{m+n-2}\,dx$$

$$= \left[\frac{x^{m+n-1}}{m+n-1}\right]_0^1$$

$$= \frac{1}{m+n-1}$$

Equation (a) becomes,

$$\beta\ \ (m,n) = \frac{(n-1)(n-2)\ldots 1}{m(m+1)\ldots(m+n-1)}$$

$$=\frac{(n-1)!(m-1)!}{(m+n-1)!}$$

$$\therefore \beta(m,n) =\ \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

Note:

In the above case, we also have,

$$\beta\,(m,n) \;=\; 2\,\frac{(2m-2)(2m-4)\dots 2}{(2m+2n-2)(2m+2n-4)\dots(2n+2)}\frac{1}{2n}$$

$$= \frac{(m-1)!}{(m+n-1)(m+n-2)\dots(n+2)}\frac{1}{n}$$

$$= \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

Example:

(1). Express $f(x) \;=\; x^m(1-x^p)^n$ from $0 \to 1$ in terms of beta function

Let $z \;=\; x^p$

Differentiating this in terms of $x,$ we get,

$px^{p-1}dx = dz$

$dx \;=\; \frac{1}{p}\frac{dz}{z^{\frac{p-1}{p}}}$

$X\!: 0 \to 1$

$Z\!: 0 \to 1$

$$\int_0^1 x^m\,(1-x^{p)})^n dx \;= \int_0^1 z^{\frac{m}{p}}\,(1-z)^n\,\frac{1}{p}\frac{dz}{z^{\frac{p-1}{p}}}$$

$$= \frac{1}{p}\int_0^1 z^{\frac{m+1}{p}-1}\,(1-z)^n dz$$

$$= \frac{1}{p}\beta(\frac{m+1}{p}, n+1)$$

Which is the required form.

(2). $\int_0^1 x^5\,(1-x^{3)})^3 dx = \frac{1}{3}\beta\left(\frac{6}{3},4\right)$

$$= \frac{1}{3}\beta(2{,}4)$$

$$= \frac{1!3!}{5!\cdot 3}$$

$$= \frac{1}{60}$$

(3). $\int_0^1 x^m(1-x^2)^n dx = \frac{1}{2}\beta(\frac{m+1}{2}, n+1)$

(4). $\int_0^1 \frac{x^2}{\sqrt{(1-x^5)}}$ dx = $\int_0^1 x^2\,(1-x^5)^{\frac{-1}{2}}$ dx

$= \frac{1}{5}\beta(\frac{3}{5}, \frac{1}{2})$

CHAPTER 3

GAMMA FUNCTION

A mathematical function known as the gamma function, or $\Gamma(p)$ extend the factorial function to complex number with the exception of negative integers, for which it is undefined. The integral is used to express it:

$$\Gamma(p) = \int_0^\infty e^{-x}\ x^{p-1}\, dx$$

Applications of this function can be found in many disciplines, such as engineering, physics, and mathematics. When dealing with issues requiring combinations, permutations, and continuous distributions, it is especially helpful.

CONVERGENCE OF GAMMA FUNCTION

If $p \geq 1$, the gamma function $\Gamma(p) = \int_0^\infty e^{-x}\ x^{p-1}\, dx$ is an improper integral of first kind and if $p < 1$, it is improper integral of third kind.

$$\Gamma(p) = \int_0^\infty e^{-x}\ x^{p-1}\, dx$$

$$= \int_0^1 e^{-x}\ x^{p-1}\, dx + \int_1^\infty e^{-x}\ x^{p-1}\, dx$$

$$= C + D$$

where,

$$C = \int_0^1 e^{-x}\ x^{p-1}\, dx$$

$$D = \int_1^\infty e^{-x}\ x^{p-1} dx$$

Case (a): $p \geq 1$

Here C is proper integral, and D is an improper integral of first kind.

Let $f(x) = e^{-x}x^{p-1}$ and $g(x) = \frac{1}{x^2}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} x^{p+1}e^{-x} = 0$$

Since,

$\int_1^\infty \frac{1}{x^2} dx$ converges, then by applying Quotient test, we get

$\int_1^\infty e^{-x} \, x^{p-1} dx$ is convergent.

That is, D is convergent.

Here $C + D$ is convergent.

$\therefore$ $\Gamma(p)$ is convergent.

Case (b): $p < 1$

Here C is an improper integral of second kind and D is an improper integral of first kind.

Let $f(x) = e^{-x} x^{p-1}$ and let $g(x) = \frac{1}{x^{1-p}}$

$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} e^{-x} = 1$

Since,

$\int_0^1 \frac{1}{x^{1-p}} dx$ converges then $1 - p < 1 (p > 0)$.

Then by applying Quotient test, we get,

$\int_0^1 x^{p-1} e^{-x} \, dx = C$, Converges when $p > 0$.

Similar to case (a) we get that,

$D = \int_1^\infty e^{-x} x^{p-1} \, dx$ converges for $p < 1$

That is, $C + D$ converges when $0 < p < 1$.

$\therefore$ $\Gamma(p)$ converges when $0 < p < 1$.

From case (a) and case (b), we get that

$\Gamma(p) = \int_0^\infty e^{-x} \, x^{p-1} \, dx$ converges for all $p > 0$.

RECURRENCE FORMULA FOR GAMMA FUNCTION

$\Gamma(p) = (p-1)\Gamma(p-1)$

$\Gamma(p) = \int_0^\infty e^{-x}\ x^{p-1}\, dx$

$= \left[x^{p-1}\frac{e^{-x}}{-1}\right]_0^\infty + \int_0^\infty (p-1)x^{p-2}\ e^{-x}\, dx$

$= (p-1)\int_0^\infty x^{\ p-2} e^{-x}\, dx$

$= (p-1)\Gamma(p-1)$

$\therefore\ \Gamma(p) = (p-1)\Gamma(p-1)$

Which is the required recurrence formula for gamma function.

Note:

If p is positive integer,

$\Gamma(p) = (p-1)\Gamma(p-1)$

$\Gamma(p) = (p-1)(p-2)\Gamma(\mathrm{p}-2)$

$\Gamma(p) = (p-1)(\mathrm{p}-2)\ldots\Gamma(1)$ ----- (1)

Where,

$\Gamma(1) = \int_0^\infty e^{-x}\ dx$

$= \left[\frac{e^{-x}}{-1}\right]_0^\infty$

$= 1$

Then equation (1) becomes,

$\Gamma(p) = (p-1)(p-2)\ldots 1$

$= (p-1)!$

$\therefore\ \Gamma(p) = (p-1)!$

RELATION BETWEEN BETA FUNCTION AND GAMMA FUNCTION

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m + n)}$$

Proof:

We have, $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

$$= 2\int_0^{\frac{\pi}{2}} sin^{\,2m-1}\theta \cos^{2n-1}\theta \; d\theta \text{ ---- (1)}$$

Also, $\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx$

Put, $x = t^2$

then, $dx = 2t\, dt$

$$x : 0 \to \infty$$

$$t : 0 \to \infty$$

$$\Gamma(n) = \int_0^\infty e^{-t^2}\, t^{2n-2}\, 2t\, dt$$

$$= 2\int_0^\infty e^{-t^2}\, t^{2n-1}\, dt \text{ ----- (2)}$$

$$\Gamma(m)\Gamma(n) = 2\int_0^\infty e^{-x^2}\, x^{2m-1} dx \cdot 2\int_0^\infty e^{-y^2}\, y^{2n-1} dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-x^2} x^{2m-1} e^{-y^2} y^{2n-1}\, dx\, dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\, dx\, dy$$

Converting into polar form,

$x = r\ cos\theta$, $y = r\ \sin\theta$

$$dxdy = r\, dr\, d\theta$$

$$r : 0 \to \infty$$

$$\theta : 0 \to \frac{\pi}{2}$$

$$= 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, r^{2m-1} (cos\theta)^{2m-1} r^{2n-1} (sin\theta)^{2n-1} r\, dr\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{\,-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta \;\, dr d\theta$$

$$= 4\left(\int_0^{\infty} e^{-r^2}\, r^{2(m+n)-1} dr\right)\, \int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta \;\, d\theta$$

$$= \left(2\int_0^{\infty} e^{-r^2}\, r^{2(m+n)-1} dr\right)\left(\, 2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta\right)$$

From (1) and (2)

$$\Gamma(m)\Gamma(n) = \Gamma(m+n)\beta(m,n)$$

$$\therefore \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Hence the proof.

$*$ Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof:

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Put, $m = n = \frac{1}{2}$

We get,

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \beta\left(\frac{1}{2},\frac{1}{2}\right)$$

$$\Gamma\left(\frac{1}{2}\right)^2 = \beta\left(\frac{1}{2},\frac{1}{2}\right)$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{\,1-1}\theta\, cos^{\,1-1}\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} d\theta$$

$$= \pi$$

$$\therefore\; \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Hence the proof.

DUPLICATION FORMULA

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$

Proof:

We have,

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \ \cos^{2n-1}\theta \ d\theta$$

Put $m = n$,

$$\beta(m,m) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \ \cos^{2m-1}\theta \ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (sin\theta cos\theta)^{2m-1} \ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta$$

$$= \frac{1}{2^{2m-2}}\int_0^{\frac{\pi}{2}} \sin^{2m-1} 2\theta \ d\theta$$

Put $2\theta = t$

$$2d\theta = dt$$

$$\theta : 0 \longrightarrow \frac{\pi}{2}$$

$$t : 0 \longrightarrow \pi$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi} \sin^{2m-1} t \ \frac{dt}{2}$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1} t \ dt$$

$$= \frac{1}{2^{2m-1}} 2\int_0^{\frac{\pi}{2}} \sin^{2m-1} t \ dt$$ [If $f(2a-x) = f(x)$ then, $\int_0^{2a} f(x) = 2\int_0^{a} f(x)$]

$$= \frac{1}{2^{2m-1}} 2\int_0^{\frac{\pi}{2}} \sin^{2m-1} t \ \cos^{0} t \, dt$$

$$\beta(m,m) = \frac{1}{2^{2m-1}}\beta\left(m, \ \frac{1}{2}\right)$$

$$\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$

Hence the proof.

CHAPTER 4
APPLICATIONS

Improper Integrals

Applications for improper integrals can be found in many fields of mathematics, science, and engineering. These include integrals across unbounded intervals and integrands with infinite discontinuities. Inappropriate integrals are frequently utilised in the following important domains:

1. Physics:
   - Calculation of physical quantities such as work, energy, electric field, and gravitational force.
   - Modeling phenomena in classical mechanics, electromagnetism, thermodynamics, and quantum mechanics.
   - Evaluating wave functions and probability amplitudes in quantum mechanics.
2. Engineering:
   - Analysis of signals and systems in electrical engineering, particularly in Fourier analysis and signal processing.
   - Calculation of moments of inertia and center of mass in mechanical engineering.
   - Design and analysis of control systems in control engineering.
   - Solution of differential equations describing physical systems.
3. Statistics and Probability:
   - Calculation of probability distributions and expected values in probability theory.
   - Estimation of parameters in statistical inference and hypothesis testing.
   - Modeling of random processes and stochastic systems.
4. Finance and Economics:
   - Valuation of financial derivatives and options using stochastic calculus.
   - Calculation of expected returns and risk measures in portfolio management.
   - Modeling of economic variables and forecasting future trends.
5. Computer Science:
   - Integration techniques are used in numerical methods for solving differential equations and optimization problems.

- Evaluation of complex algorithms involving probability distributions and statistical analysis.
- Development of machine learning algorithms for pattern recognition and data analysis.

6. Geosciences and Environmental Science:
   - Analysis of spatial and temporal variations in environmental data.
   - Modeling of fluid flow, heat transfer, and chemical transport in Earth systems.
   - Evaluation of seismic signals and interpretation of geophysical data.

Overall, improper integrals provide powerful mathematical tools for analyzing real-world phenomena, modeling complex systems, and making predictions in different fields of study. Their applications are broad and interdisciplinary, making them indispensable in both theoretical and practical aspects.

Beta Function:

Another significant special function in mathematics with a wide range of applications is the beta function, represented by the symbol B(x,y). Here are a few noteworthy uses:

1. Probability and Statistics:

- The beta distribution, which is derived from the beta function, is widely used in Bayesian statistics to model random variables with values between 0 and 1. It's particularly useful in modeling proportions, rates, and probabilities.
- The beta function is used in defining the beta distribution's probability density function (PDF) and cumulative distribution function (CDF), which are essential for statistical inference and hypothesis testing.

2. Integral Calculus:

- The beta function arises naturally in evaluating certain types of definite integrals, particularly those involving trigonometric functions and powers of sine and cosine.
- It is employed in evaluating multivariate integrals, especially in polar, cylindrical, and spherical coordinate systems.

3. Engineering and Physics:

- In physics, the beta function appears in the context of nuclear decay processes, such as beta decay, where it describes the probability distribution of the energy of emitted beta particles.
- It is used in fluid dynamics to calculate various fluid flow properties, such as the velocity profile in laminar flow between parallel plates.
- In control theory and system dynamics, the beta function is utilized in modeling transfer functions and analyzing stability criteria for dynamic systems.

4. Finance and Economics:

- The beta function finds applications in finance for modeling asset pricing models, such as the Capital Asset Pricing Model (CAPM), where it represents the covariance between an asset's returns and the market's returns.
- It is used in risk management to quantify the relationship between the risk of an individual asset or portfolio and the overall market risk.

5. Machine Learning and Data Analysis:

- The beta distribution, derived from the beta function, is used in Bayesian inference for modeling prior and posterior distributions of parameters in machine learning algorithms.
- It is employed in modeling the probability of success in binary classification tasks and estimating parameters in logistic regression models.

6. Chemistry and Biology:

- In chemistry, the beta function is used in calculating the molecular partition functions and in describing the distribution of molecular energies.
- In biology, it can be used in modeling population growth rates and in the analysis of genetic data, particularly in Bayesian estimation of allele frequencies.

Gamma Function:

The gamma function, denoted by $\Gamma(z)$, has numerous applications across mathematics, science, and engineering. Some key applications include:

1. Combinatorics and Probability:

- The gamma function is deeply connected to combinatorial analysis, particularly in counting permutations and combinations. It appears in formulas for binomial coefficients and multinomial coefficients.
- In probability theory, the gamma function is used to define the gamma distribution, which models the time until an event occurs in processes such as radioactive decay and queueing systems.

2. Analysis and Integral Calculus:

- The gamma function generalizes the factorial function to real and complex numbers (except negative integers). It is used extensively in evaluating integrals, especially those involving powers of trigonometric functions, exponential functions, and polynomials.
- It appears in various integral transforms, such as the Laplace transform, Mellin transform, and Fourier-Bessel transform.

3. Number Theory:

- The gamma function is involved in the study of special values of certain Dirichlet series and Euler products.
- It appears in formulas related to the Riemann zeta function, including the functional equation of the zeta function and its analytic continuation.

4. Physics and Engineering:

- In physics, the gamma function arises in quantum mechanics, particularly in calculating probabilities and amplitudes of quantum processes.
- It is used in engineering disciplines, such as signal processing, control theory, and fluid dynamics, for solving differential equations and analyzing complex systems.

5. Statistical Distributions:

- The gamma function is used to define several important probability distributions, including the gamma distribution, chi-square distribution, and exponential distribution.
- These distributions are widely used in statistical modeling and data analysis, for example, in reliability analysis, survival analysis, and queuing theory.

6. Special Functions and Mathematical Identities:

- The gamma function is a fundamental building block for many other special functions, such as the beta function, hypergeometric function, and confluent hypergeometric function.
- It appears in various mathematical identities and series representations, including Euler's reflection formula, Gauss's multiplication theorem, and Stirling's approximation for the factorial function.

CONCLUSION

The beta and gamma functions are incredibly important in mathematics, physics, engineering, and various other fields. In essence, the beta and gamma functions are powerful mathematical tools that provide solutions to a wide range of problems in various fields, making them indispensable in both theoretical analysis and practical applications.

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