A STUDY ON GRACEFUL GRAPHS

Dissertation submitted in the partial fulfilment of the requirement for the

MASTER'S DEGREE IN MATHEMATICS

By

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DECLARATION

I, Nivia Joy, hereby declare that this project entitled **'A STUDY ON GRACEFUL GRAPHS'** is a bonafide record of work done by me under the guidance of Dr. Seethu Varghese, Assistant professor, Department of Mathematics, Bharata Mata College, Thrikkakara and this work has not previously formed by the basis for the award of any academic qualification, fellowship or other similar title of any other University or Board.

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CERTIFICATE

This is to certify that the project entitled **A STUDY ON GRACEFUL GRAPHS** submitted for the partial fulfilment requirement of Master's Degree in Mathematics is the original work done by Nivia Joy during the period of the study in the Department of Mathematics, Bharata Mata College, Thrikkakara under my guidance and has not been included in any other project submitted previously for the award of any degree.

Dr. Seethu Varghese
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1:
2:

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CONTENTS

INTRODUCTION

One of the best known labeling methods of graphs, *graceful labeling* was introduced in 1967 by the mathematician Alexander Rosa. It was firstly named as *β labeling.* Later, American mathematician Golumb renamed it as *graceful* in 1972. Consider the graph G = (V,E) of m vertices and n edges. Then the 1-1 mapping Ψ from vertex set V(G) into the set $\{0, 1, 2, .., n\}$ is called the graceful labeling of graph G. With this, if we define, for any edge e = uv $\in$ E(G), the value Ψ^*(e) = |Ψ(u) - Ψ(v)|. Then Ψ^* is a one-to-one mapping of the set E(G) onto the set $\{1, 2, ..., n\}$. If a graph has graceful labeling, it is said to be graceful[4].

Rosa established a parity criterion for the straightforward graph G with n edges. He demonstrates that G is not graceful if every vertex has an even degree and n $\equiv$ 1, 2 (mod 4). Later Golumb confirmed it as a necessary condition that, if G is graceful, even (simple) graphs with n edges then necessarily [(n+1)/2] $\equiv$ 0 (mod 2) [12].

Although utmost graphs aren't graceful, graphs that have some kind of chronicity of structure are graceful. Numerous variations of graceful labeling have been presented in recent times by experimenters. All cycles C_n are graceful iff n $\equiv$ 0 or 3 (mod 4). All paths P_n, wheels W_n and complete bipartite graphs K_{mn} are graceful. The complete graphs K_n are graceful iff n $\leq$ 4. It has been conjectured that all trees are graceful and it is still an open problem.

It has wide range of application in other fields such as in coding theory, communication networks, dental arch etc. Also more than 400 papers have been published on the subject of graph labeling.The graceful labeling problem is to find out whether a graph is graceful. This project is divided into 5 chapters.

In the Chapter 1, we deal with the basic definitions of graph theory. The formal definition of graceful labeling of a graph, the gracefulness of specific graph classes, and some general conclusions about graceful labeling of graphs are presented in Chapter 2. In Chapter 3, we focus on findings related to the Graceful Tree Conjecture and various methods to challenge the conjecture. In chapter 4, we discuss the variations of graceful labeling. And in Chapter 5, we discuss the applications of graceful graphs in various fields.

CHAPTER 1

SOME BASIC CONCEPTS OF GRAPH THEORY

Definition1.1: A graph is an ordered triple G = (V(G), E(G), I_g) where V(G) is a non empty set, E(G) and V(G) are disjoint, and I_g is an incidence relation that related to each element of E(G), an unordered pair of elements (same or distinct) of V(G). Elements of V(G) are called the vertices of G and elements of E(G) are called the edges of G .V(G) and E(G) are the vertex set and edge set of G independently. For the edge e of G, I_g(e) =$\{u, v\}$, we write I_g(e) = uv [12].

Example 1.1: If V(G) = $\{v_1, v_2, v_3\}$, E(G) = $\{e_1, e_2, e_3, e_4, e_5\}$ and I_g is given by $I_g(e_1) = \{v_1, v_2\}$, $I_g(e_2) = \{v_1, v_2\}$, $I_g(e_3) = \{v_1, v_3\}$, $I_g(e_4) = \{v_2, v_3\}$, $I_g(e_5) = \{v_3, v_3\}$, then (V(G), E(G), I_g) is a graph.

The diagrammatic representation of the graph is given below.

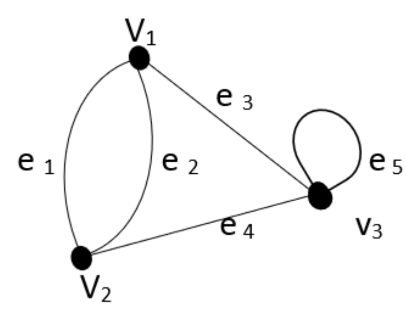


Figure 1.1

Definition 1.2:The vertices u and v are known as the *end vertices or ends* of the edge e if I_G (e) = $\{u, v\}$. In this scenario, we say that e is incident with each of its ends. Each edge is said to join its ends. Also vertices u and v are incident with e. A collection of two or more graph edges If they share the same pair of unique ends, then G is referred to as a set of *multiple or parallel edges.* We write e = uv if e is an edge with the end vertices u and v. A *loop* at the common vertex is the edge whose two ends are identical. If uv is an edge of G then the vertex u is a neighbor of v in G, u ≠ v. The *open neighborhood* of v, also known as the neighbor set of v, is indicated by the set N(v), whereas the *closed neighborhood* of v in G is denoted by the set N[v] = N(v) U v. These open and closed neighborhoods of v are indicated by N_G(v) and N_G[v], respectively, when G needs to be made explicit. If there is an edge of G with u and v as its ends, then Vertices u and v are *adjacent* to each other in G. Similarly two distinct edges e and f are said to be *adjacent* if and only if they have a common end vertex. A graph having no loops and parallel edges are called *simple graph* [1].

Example 1.2: In figure 1.1, edge $e_3 = v_1v_3$, edges e_1 and e_2 are parallel edges and e_5 is a loop at v_3; N(v_3) = $\{v_1, v_2\}$, N[v_3] = $\{v_1, v_2, v_3\}$. Further, v_1 and v_3 are adjacent vertices and e_3 and e_4 are adjacent edges.

Definition 1.3: A graph in which both V(G) and E(G) are finite is called a *finite graph* and those which are not finite is called *infinte graph.*

Definition 1.4: If all n vertices of graph G distinguished from one another by labels $v_1, v_2, ..., v_n$,then G is a labelled graph [1].

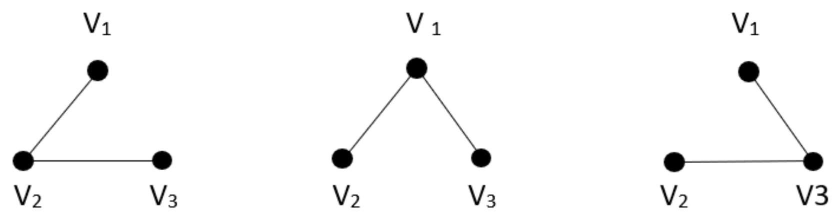


Figure 1.2

Definition 1.5: If every pair of distinct vertices of a simple graph G are adjacent in G, then the graph G is called the *complete graph* K_n.

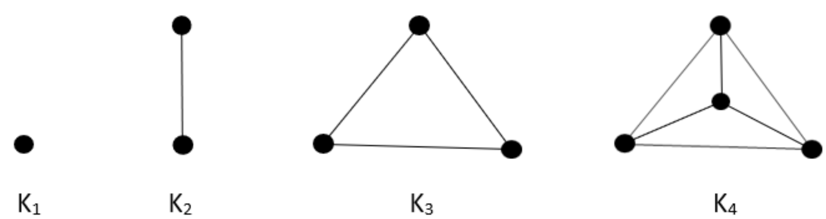


Figure 1.3

Definition 1.6: A graph with its vertex set is a singleton and containing no edges is called a*trivial graph.*Also a graph in which the vertex set can be divided into two non empty subsets X and Y whose each edges has one end in X and other end in Y is called a *bipartite graph* and the pair(X,Y) is called the *bipartition* of the bipartite graph denoted by G(X,Y). Here if each vertex of X is adjacent to all the vertices of Y, then G(X,Y) is said to be *complete bipartite graph.* A complete bipartite graph G(X,Y) with $|X|=$ P and $|Y|=$ q is denoted by $K_{p,q}$ [1].

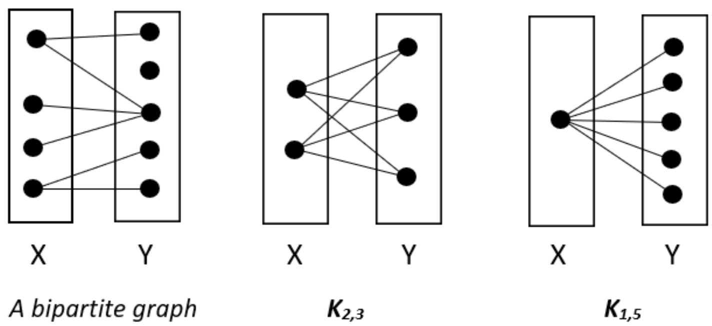


Figure 1.4

Definition 1.7: A graph whose vertices can be divided into k distinct sets, none of which contain adjacent vertices, is referred to as a *k-partite graph*[1].

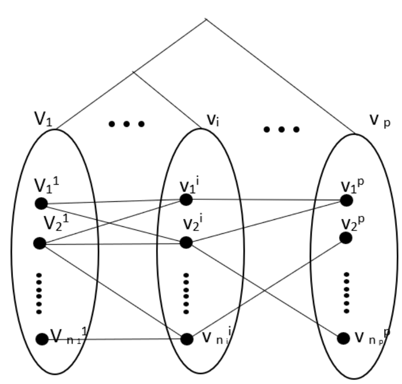


Figure 1.5

Definition 1.8: A k-partite graph is said to be *complete* if there is an edge connecting each pair of vertices from several independent sets.

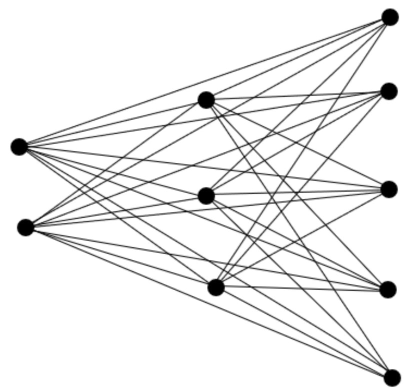


Figure 1.6

Definition 1.9: Let H be a graph. If V(H) $\subseteq$ V(G), E(H) $\subseteq$ E(G) and I_H is the restriction of E(H), then graph H is called a *subgraph* of G. Here G is called the supergraph of H. If either V(H) $\neq$ V(G) or E(H) $\neq$ E(G), then the subgraph H of graph G is a *proper subgraph.* If every edge of G with ends in V(H) is also an edge of a subgraph H of G, then the subgraph is said to be *induced* and if V(H) = V(G), then a subgraph H of G is a *spanning subgraph* of G. If S $\subseteq$ V(G), then the subgraph induced by S of G is denoted by G[S] [1].

Definition 1.10: Let G be a graph and v $\in$ V(G). The *degree* of v in G is the number of edges incident at v in G and is denoted by d_G(v) or d(v). While finding the degree of v, a loop at v is counted twice. δ(G) and Δ(G) denotes the minimum and maximum degrees of vertices of G respectively. If every vertex of graph G has a degree k, then the graph is said to be *k-regular* and a graph is called *regular* if it is k-regular for some non-negative integer k [1].

Definition 1.11: An *isolated vertex* of graph G is the vertex with degree 0. Also *pendant vertex* is the vertex with degree 1 and *pendant edge* is the edge that the pendant vertex makes with its adjacent vertex. *Degree sequence* of G is obtained when the degrees of the vertices of G are noted as a sequence, where the vertices are taken in the same order[1].

Definition 1.12: An alternating sequence W: $v_0e_1v_1e_2v_2....e_pv_p$ of vertices and edges, begining and ending with vertices is called a *walk* in graph G. Here the walk W is also referred as v_0-v_p walk. And a walk is said to be *closed* if $v_0 = v_p$, otherwise it is an *open walk*. If all the edges in the walk are distinct, then the walk is called a *trail* and it is called a *path* if all the vertices are distinct. Thus a path is always a trail. But the converse is not true. A closed trail in which in the vertices are all distinct is called a *cycle*. The number of the edges in W is called the length of W and d(u,v) denotes the length of the shortest u-v path in G, which is the distance between the vertices u and v in G [1].

Example 1.3: In figure 1.7, $v_6e_8v_1e_7v_5e_6v_4e_5v_1e_8v_6$ is a walk but not a trail (since edge e_8 is repeated). $v_6e_9v_5e_6v_4e_5v_1e_1v_2e_2v_3$ is a trial which is also a path and $v_1e_8v_6e_9v_5e_6v_4e_5v_1$ is a cycle.

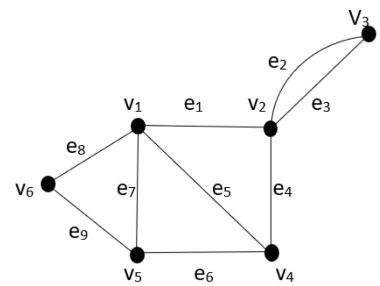


Figure 1.7

Definition 1.13: A cycle of length k is denoted by C_k and a path on k vertices is denoted by P_k. In particular, C_3 is referred as triangle, C_4 as a square and C_5 as a pentagon.

Definition 1.14: For two vertices u, v $\in$ G, if there exist a u-v path in G, then graph G is *connected.* Also it is an equivalence relation on V(G). Let $V_1, V_2, ..., V_w$ be the equivalence classes. Then the subgraphs G[V_1], G[V_2],....,G[V_W] are called the components of G. Graph G is connected, if w $=$ 1 ; otherwise G is disconnected [1].

Definition 1.15: A spanning trail in a graph G that contain all the edges of G is called an *Euler trail* and a closed euler trail in G is called an *Euler tour.* If G has an euler tour, then G is called *Eulerian* [1].

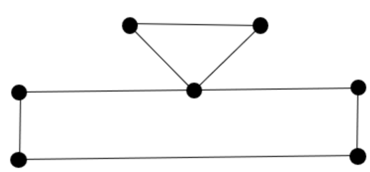


Figure 1.8

Definition 1.16: A connected graph without cycles is defined as a *tree*. A *forest* is a graph with trees as its connected elements. A leaf is any vertex of degree 1 in a tree [1].

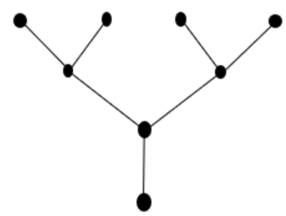


Figure 1.9

Definition 1.17: Let G be a connected graph.

1. The *diameter* of G is defined as:

$$\max\{d(u,v)/u,v \in \mathrm{V(G)}\}$$

is denoted by diam(G).

2. If v is a vertex of G, then its *eccentricity* e(v) is defined by:

$$\mathrm{e(v)} = \max\{d(v,u)/u \in \mathrm{V(G)}\}$$

3. The *radius* r(G) of G is the minimum eccentricity of G.

$$\mathrm{r(G)} = \min\{e(v)/v \in \mathrm{V(G)}\}$$

Also

$$\mathrm{diam(G)} = \max\{e(v)/v \in \mathrm{V(G)}\}$$

4. If e(v) = r(G), then the vertex v of G is called the *central vertex* of G. The set of central vertices of G is called the *center* of G [1].

CHAPTER 2

INTRODUCTION TO GRACEFUL GRAPHS

Graph Labeling

Graph labeling, also known as valuation of a graph, is a map that assigns values so that a property is satisfied. It can be given either to vertices or edges or to both. If we give labels to the vertices of graph G , then it is called *vertex labeling* and is called *edge labeling* if edges of G are labeled. If both edges and vertices are labeled, then it is called *total labeling*.

Graceful Labeling

Definition: Consider the graph G with m vertices and n edges. Then the 1-1 mapping Ψ from vertex set V(G) into the set $\{0, 1, 2, .., n\}$ is called the **graceful labeling** of graph G. With this, if we define, for any edge e = uv ∈ E(G), the value Ψ*(e) = |Ψ(u) - Ψ(v)|. Then Ψ* is a one-to-one mapping of the set E(G) onto the set $\{1, 2, ..., n\}$. If a graph has graceful labeling, it is said to be **graceful** [12].

Although elegant labeling has been studied over 50 years, not many general conclusions have been reached. Since it is sufficient to display a graceful labeling for each graph in the class, the majority of the results centered on demonstrating the gracefulness of a graph class. On the other hand, results on a graph's lack of grace rely mostly on a necessary condition that is only true for Eulerian graphs or on attempts to name the graph graciously until a contradiction, which is generally ineffective [12].

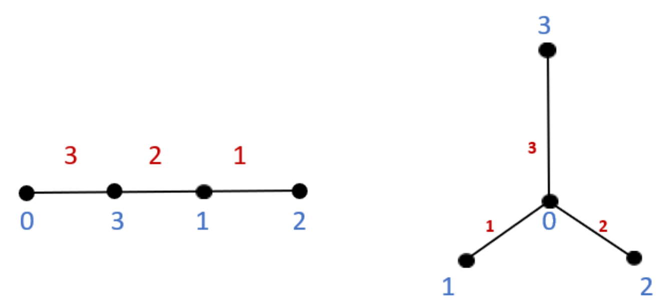


Figure 2.1: Graceful labeling of P_4 and $K_{1,3}$

Theorem 2.1: The path graph P_m is graceful for all m≥1 [12].

Proof: Consider the path graph P_m. Let V(P_m) = $\{u_0, u_1, u_2, ..., u_{m-1}\}$ be the vertex set of P_m such that $u_{k-1}u_k \in$ E(P_m) and |E(P_m)| = m-1 edges. Label the vertices with integers from 0 to m-1 such that each number between 1 and m-1 becomes an edge label. Initiate with edge label m-1. It is possible to give an absolute difference m-1 only in one way. i.e, putting a vertex with label 0 next to a vertex with label m-1. For that, let's label $\Psi(u_0)$ = 0 and $\Psi(u_1)$ = m-1. Now to get an edge label m-2, there are only two ways. i.e. m-2 = | (m-2) - 0 | = | (m-1) - 1 |.

Since it is path graph, u_0 has no more neighbouring vertices that are not labeled. So we obtain the edge label m-2 by fixing $\Psi(u_2)$ = 1. Continuing this, our final labeled path has labeling as follows:

$$\Psi(u_k) = \begin{cases} \frac{k}{2} & ; if\, k\, is\, even \\ m - \frac{(k+1)}{2} & ; if\, k\, is\, odd \end{cases}$$

Next we show that Ψ is a graceful labeling of P_m. For that we've to show that the edge label 1 occur on the last edge $u_{m-2}u_{m-1}$. $\Psi(u_{m-1})$ = m/2 and $\Psi(u_{m-2})$ = (m-2)/2, if m is even. Therefore,

$$\Psi^*(u_{m-1}u_{m-2}) = \text{m/2 - (m-2)/2} = 1$$

Hence the theorem. $\diamond$

Theorem 2.2: If G = (V,E), |V| = p is graceful, then there is a partition of V into disjoint subsets V_1 and V_2 such that the number of edges having one end in V_1 and the other in V_2 is the least integer greater than or equal to p/2 [12].

Proof: Given graph G = (V, E) of order p has a graceful labeling Ψ. Consider the partition P=(V_1, V_2) of V such that $V_1 = \{u \in$ V ; Ψ(u) $\equiv$ 0 (mod 2)$\}$.

Then there are least integer greater than or equal to p/2 odd values between 1 and p, and we get an odd difference only from the substraction between an odd and an even number. Therefore the number of edges having one end in V_1 and the other in V_2 is the least integer greater than or equal to p/2 [12]. ⋄

Theorem 2.3: Let G be an Eulerian graph m vertices. If m $\equiv$ 1, 2 (mod 4), then G is not graceful [12].

Proof: Assume that m $\equiv$ 1, 2 (mod 4). Suppose that G = (V, E) is a graceful Eulerian graph. Let Ψ : V to [0, m] be a graceful labeling of G and consider the eulerian cycle C = $(u_0, u_1, u_2, ..., u_{m-1}, u_m = u_0)$ of G. Taking the sum of the edge labels of C modulo 2, we have

$\sum_{i=1}^{m} \Psi^*(u_{i-1}u_i) = \sum_{i=1}^{m} |\Psi(u_{i-1})$ - $\Psi(u_i)|$
$= \sum_{i=1}^{m} \Psi(u_{i-1})$ - $\Psi(u_i)$
$\equiv 0$ (mod2)

All the edges on the cycle C are distinct. since Ψ is a graceful labeling of G, we've:

$$\sum_{e} \in_{E} \Psi^*(\mathrm{e}) = \sum_{k=1}^{m} k = \frac{m(m+1)}{2} \equiv 0 \ (\mathrm{mod}2)$$

Thus, we must have m $\equiv$ 0, 3 (mod 4) in order to satisfy the above equation which is a contradiction to our assumption. Therefore G is ungraceful[12]. ⋄

Theorem 2.4: Every graph is an induced subgraph of a graceful graph [12].

Proof: Consider the m - vertex graph G = (V,E). Create a graph H from G such that G is an induced subgraph of H and that graph H is graceful. Consider the vertex labeling Ψ : V to [0,k], for some k $\geq$ m and the edge labeling Ψ* which are injective. Now consider Ψ(u) = 0 and Ψ(v) = k ; u, v $\in$ V(G). It lacks certain edge labels since it is not graceful. Let the collection of omitted edge labels be $\{x_1, x_2, x_3..., x_r\}$. Assume that $x_{s+1}, x_{s+2}, x_{s+2}, ..., x_r$ are the vertex labels in that set while $x_1, x_2, ..., x_s$ are not. Now add a new vertex w_i with $\Psi(w_i) = x_i$;1 $\leq$ i $\leq$ s and connect w_i to u so that Ψ*(uw_i) = x_i.Also add another vertex W_i with $\Psi(W_i)$ = k+x_i ; s+1$\leq$ i $\leq$ r and connect vertex W_i to u and v so that Ψ* (u W_i) = k + x_i and Ψ* (v W_i) = x_i. Due to the creation of vertex labels with values bigger than k, the previous step may have resulted in the insertion of further missing edge labels.The missing edge labels in question are not vertex labels, though. To fix this, add a new vertex z_y with $\Psi(w_i)$ for each additional missing edge label y. Connect z_y to u so that Ψ*(uz_y) = y. Then the final graph H is graceful and it contains G as an induced subgraph [12]. ◇

By Theorem 2.4, the non-gracefulness of graph G for graphs for which G is an Induced subgraph is irrelevant. Additionally, it asserts that any graph may always be transformed into a graceful graph.

Other Graceful Graphs

In this section, we'll show how several graph classes behave gracefully.

The majority of findings supporting gracefulness of graph classes are provided by explicit graceful labeling. There aren't many tools to deal with a graph class's lack of grace. In essence, we just have Theorem 2.2 and Theorem 2.3. Making an attempt to label the graph and finding a contradiction is another technique to make a point.

Theorem 2.5: The *complete graph* K_n is graceful if and only if n $\leq$ 4 [12].

Proof: A graph having graceful labeling is provided. The resulting labeling is also graceful if we shift every vertex label from k to m - k because the edge labels will remain the same. Specifically, m-a and m-b are created from the end vertices of an edge with labels a and b and |(m-a) - (m-b)|. This is called the complementarity property.

Now, for K_n with n $>$ 4, as previously, in order to obtain the edge label m, we need a vertex with label 0 next to a vertex with label m. However, here every vertex is next to every other vertex. So, without losing generality, we can label any vertex with 0 and any other vertex with m. To obtain the edge label m - 1, we have two possibilities : m - 1 $=$ |(m - 1) - 0| $=$ |m - 1|.The complementarity quality, however, enables us to select any one without losing generality. We obtain the edge labels 1 and m - 1 by choosing to label a vertex with 1. Obtaining the edge label m - 2$=$ |(m - 2) - 0| $=$ |(m - 1) - 1| $=$ |m - 2|. m - 1 or 2 would result in a duplicate edge label, hence we are unable to label a vertex with these values. Therefore, the only option we have is to assign a vertex the label m - 2, so acquiring the edge labels 2, m - 3, and m - 2.

The next edge label that has to be acquired is m - 4 because m - 3 has already been found on an edge. m - 4 = |(m-4) - 0| = |(m-3) - 1| = |(m-2) -2| = |(m-1) -3| = |m - 4|. Once more, the only option left is to label a vertex with 4, acquiring edge labels 3, 4, m - 6, and m - 4. This will avoid the need to create a duplicate edge label. Five vertices have been assigned labels at this time.However, m - 6 = 4 would be a duplicate edge label for K_5 anyway. For n $\geq$ 6, the following edge label is m - 5. However, each of the five approaches to obtain m – 5 results in a duplicate edge label. As a result, the label m - 5 cannot be on an edge and the statement is true [12]. $\diamond$

Theorem 2.6: The *cycle graph* C_n is graceful if and only if n $\equiv$ 0,3 (mod 4) [12].

Proof: Cycle graphs are eulerian graphs. In eulerian graphs, if n $\equiv$ 1, 2 (mod 4), then C_n is not graceful. If not, let V(C_n) = $\{v_1, v_2, ..., v_{n-1}, v_n\}$. If n $\equiv$ 0 (mod 4), then label the vertices according to the formula:

$$\Psi(v_i) = \begin{cases} \frac{i-1}{2} & ; i\ odd \\ n+1-\frac{i}{2} & ; i\ even\ and\ i\ \leq \frac{n}{2} \\ n-\frac{i}{2} & ; i\ even\ and\ i\ > \frac{n}{2} \end{cases}$$

If n $\equiv$ 3 (mod 4), then label v(C_n) as follows:

$$\Psi(v_i) = \begin{cases} \frac{i-1}{2} & ;\, i\ odd \\ n+1-\frac{i}{2} & ;\, i\ even\, and\, i\ \leq \frac{n+1}{2} \\ n-\frac{i}{2} & ;\, i\ even\ and\, i\ > \frac{n+1}{2} \end{cases} \qquad \diamond$$

Theorem 2.7: The *wheel graph* W_p is graceful for all p $\geq$ 3 [12].

Proof: Let V(W_p) $= \{u_0, u_1, ..., u_{p-1}, v\}$ where v is the vertex joined with the cycle and consider the following instances:

If p $\equiv$ 0 (mod 2), then the following formula gives a graceful labeling:

$$\Psi(u_i) = \begin{cases} 2p & ;\, i\ =\ 0 \\ 2 & ;\, i\ =\ p-1 \\ i & ;\, i\ =\ 1,3,5,...,p-3 \\ 2p-i-1 & ;\, i\ =\ 2,4,6,....,p-2 \end{cases}$$

Ψ(v) $=$ 0

If p $\equiv$ 1 (mod 2), then the following formula gives a graceful labeling:

$$\Psi(u_i) = \begin{cases} 2p & ; i = 0 \\ 2 & ; i = 1 \\ p+i & ; i = 2,4,6,...,p-1 \\ p+1-i & ; i = 3,5,7,....,p-2 \end{cases}$$

$\Psi(\text{v}) = 0$ $\diamond$

Theorem 2.8: All *caterpillar* trees are graceful [12].

Proof: A *caterpillar* is a tree in which the removal of all leaves results in a path graphs. Create a planar bipartite representation of the caterpillar tree and label it as in Figure 2.2. Verifying that such a drawing approach is always feasible is simple [12]. $\diamond$

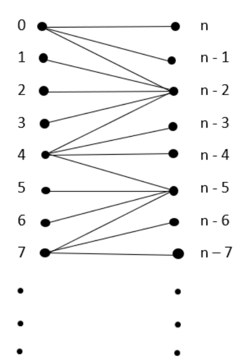


Figure 2.2: Graceful labeling of caterpillar tree

So path graph P_n is also a caterpillar tree. Also when applied to a path graph the labeling scheme given by Theorem 2.8 produces the same labeling as before.

Theorem 2.9: The *complete bipartite graph* $K_{p,q}$ is graceful $\forall$ p,q $\geq$1 [12].

Proof: Let G = (A, B, E) be a bipartite graph with a = |A| and b = |B|. Give A's vertices the numbers 0, 1, 2,.., a-1, and B's vertices the numbers a, 2a,..., ba. Every integer between 1 and ab has a distinct representation in this fashion as the difference between a number in B and a number in A [12]. $\diamond$

The idea of a bipartite graph can be reduced to a multipartite graph, and in the same way, we have the whole multipartite graph. The following theorem regarding the gracefulness of full multipartite graphs was established.

Theorem 2.10: The *complete multipartite graphs* $K_{p,q}$, $K_{1,p,q}$, $K_{2,p,q}$ and $K_{1,1,p,q}$ are graceful [12].

Additionally, Beutner hypothesized that they are the only graceful complete multipartite graphs, and computational proof was shown that this is true for **all complete multipartite graphs with at most 23 vertices**.

CHAPTER 3
GRACEFUL TREES

Today, the Graceful Tree Conjecture is still an open question, and scholars have tried a number of various methods to support the conjecture. This section contains several findings regarding the grace of trees and various approaches to the topic.

Conjecture 3.1 (Graceful tree conjecture): *Every tree is graceful.*

In chapter 2 we've already discussed that paths and caterpillars are graceful. An approach to caterpillars extended to subclasses of trees such as *spider tree, lobster tree* etc. However the characterization of lobster tree has not yet been completed. The gracefulness of all lobsters is still remain as a conjecture. In this chapter we give an approach to the gracefulness of trees [9].

Definition 3.1: A tree with atmost one vertex of degree greater than 2 is called a *spider tree* and that unique vertex having that property is called the *branch point* of the tree. And *lobster tree* is a tree with the property that the removal of all its leaves result in caterpillar.

Lemma 3.1: Let tree T has a graceful labeling Ψ and let u $\in$ V(T) with Ψ(u) = 0. If T' is the tree created from T by adding a new vertex solely adjacent to u, then T' is graceful [12].

Proof: Given u $\in$ v(T) such that Ψ(u) = 0. Let the number of edges of T be m. Then the vertex labeling Ψ' is such that $\Psi' /_{v(T)} = \Psi$ and Ψ'(v) = m+1 is a graceful labeling of T [12]. $\diamond$

Corollary 3.1.1: If w $\in$ V(T) and has label m, then a graceful tree is also produced when a new vertex that is only adjacent to w is added [12].

Proof: Take a complementary graceful labeling of Ψ. $\diamond$

Corollary 3.1.2: If H is a caterpillar tree u $\in$ V(T) has label 0 (or m), then adding an edge between u and a vertex of H with maximum eccentricity also produces a graceful tree [12].

Proof: Apply lemma 3.1, prefers to add leaves first, whenever it is reasonable to do so. The conclusion holds true for every graceful graph G as long as u $\in$ V(G) has label 0 or m, it should be noted [12]. $\diamond$

By adding a vertex to smaller graphs, we can create new graceful graphs according to Lemma 3.1. Then, it makes sense to wonder if this could be used to demonstrate the Graceful Tree Conjecture, which states that for any tree, starting from a single vertex, there exists a finite succession of graceful trees each of which is composed of the preceeding tree in the sequence added with a vertex, with the final tree in the succession being the desired tree itself [12].

Every tree must permit a graceful labeling in which every vertex may be given the label 0, in order for such a series to occur. Such graphs are known as O-rotatable graceful graphs in a broader sense. But it's not accurate to say that all trees have graceful O-rotation [12].

Let e = uv be an edge of tree T. The subtree of T including v obtained by deleting the edge uv is denoted by $T_{u,v}$. If S = $\{w \in$ V(T) : v $\in uw - path\}$, then $T_{u,v}$ = T[S] [12].

Lemma 3.2: Let u $\in$ V(T) be a vertex adjacent to u_1 and u_2 in a graceful tree T. Consider T' = T - (V($T_{u,u1}$) $\cup$ V($T_{u,u\,2}$)) and let v $\in$ V(T') , v $\neq$ u

(a) If $u_1 \neq u_2$ and $\Psi(u_1) + \Psi(u_2) = \Psi$(u) + Ψ(v), then the tree obtained by a disjoint union of T', $T_{u,u1}$, $T_{u,u2}$, and connecting v to u_1 and u_2 is graceful with the same graceful labeling Ψ.

(b) If $u_1 = u_2$ and $2\Psi(u_1) = \Psi$(u) + Ψ(v), then the tree obtained by a disjoint union of T', $T_{u,u1}$ and connecting v to u_1 produces the very graceful labeling Ψ [12].

Proof: It is sufficient to show that edge labels of uu_1 and uu_2 is same as that of vu_1 and vu_2.

(a) $|\Psi(u_1)$ - Ψ(u)$|$ = $|\Psi$(u) + Ψ(v) - $\Psi(u_2)$ - Ψ(u)$|$ = $|\Psi$(v) - $\Psi(u_2)|$
$|\Psi(u_2)$ - Ψ(u)$|$ = $|\Psi$(u) + Ψ(v) - $\Psi(u_1)$ - Ψ(u)$|$ = $|\Psi$(v) - $\Psi(u_1)|$

(b) $|\Psi(u_1)$ - Ψ(u)$|$ = $|$ ((Ψ(u)+Ψ(v))/2) - Ψ(u)$|$ = $|$(Ψ(u) - Ψ(v))/2$|$
$|\Psi(u_1)$ - Ψ(v)$|$ = $|$ ((Ψ(u)+Ψ(v))/2) - Ψ(v)$|$ = $|$(Ψ(v) - Ψ(u))/2$|$ [12]. $\diamond$

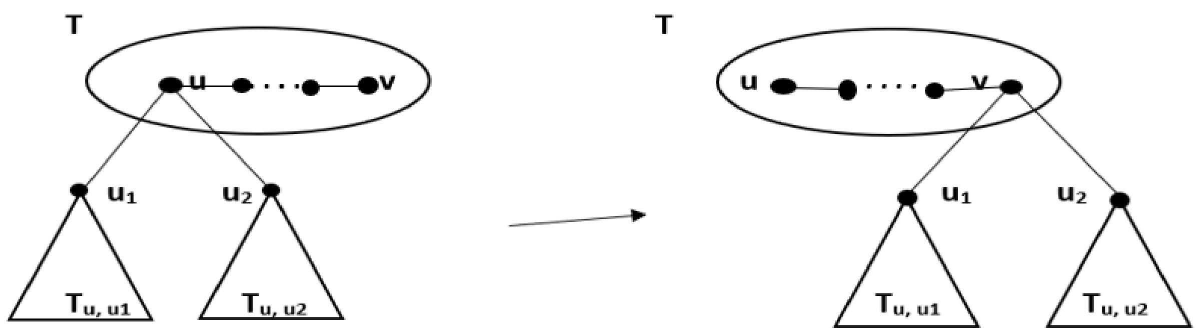


Figure 3.1: Transfer of subtrees from u to v

This operation is called the *transfer* and seen in transferring leaves from one vertex to another. For example, consider the star graph $K_{1,m}$. Let us transfer some leaves from vertex 0 to vertex m. Also in another one we can transfer k and m-k from 0 to m since k + (m-k) = 0 + m. Here the transfer from vertex u to vertex u is denoted as u$\longrightarrow$v.

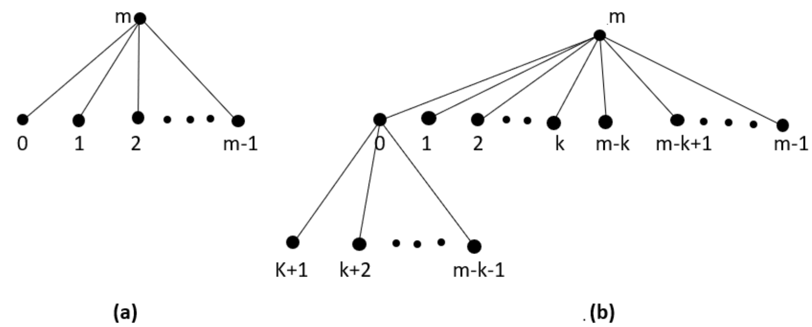


Figure 3.2 : Transfer of leaves from m to 0

Theorem 3.1: All trees with diameter 4 are graceful [12].

Proof: Consider the following transfers:
If the leaves being transferred are k, k+1, k+2, ..., k+s, it is called type 1 u $\longrightarrow$ v transfer. Here u + v = k + (k + s). This transfer leaves an odd number of vertices connected to u.

If the transferred leaves are k, k+1, ..., k+s and l, l + 1, ..., l + s with k + s < l, such a u$\longrightarrow$v transfer is called type 2 transfer. Here u + v = k + (l + s). This transfer leaves an even number of vertices connected to u.

Using Lemma 3.1, it is suffices to show that every tree T of diameter 4 with an odd-degree center vertex (which is distinctive in tree T) has a graceful labeling, with the central vertex having the most labels. This is true because any subtree rooted at a child of the central vertex in a tree with a diameter of four is a caterpillar tree.

The number of vertices adjacent to w with an even degree should be x, while those with an odd degree larger than 1 should be y. Let w be the center vertex of T. Let d(w) = 2k + 1 and have a look at Figure 3.2.b. We may derive T from that tree by following the sequence of transfers 0⟶m-1⟶1⟶m-2⟶2⟶m-3 .....; where the initial x transfer (or x-1 if y = 0) are of type 1 and the succeeding y-1 transfers (if y>1) are of type 2. .

Let us look over the first transfer to confirm that the sequence works. Let the vertices adjacent to w with even degree be $\{u_1, u_2, u_3, ..., u_x\}$. Consider fig 3.2.b the vertex with label m is w. begin the transfer from 0⟶m-1. Assume that u_1 is the vertex 0 in this case and we want to leave d(u_1) - 1 vertices associated to it. At first the vertices adjacent to 0 were k+1, k+2,....,m-k-2, m-k-1 . We can leave d(u_1)-1 vertices by type 1 transfer of a continous sequence of vertices to m-1 since 0 + m-1 = (k+1) + (m-k-2). Continuing the analysis we can see that the sequence works [12]. ⋄

Theorem 3.2: All trees with diameter 5 are graceful [12].
The proof of this also done by the transfer operation as above. ⋄

The gracefulness of trees with at most 29 vertices was proved in 2003 by Horton. So far we have **trees with at most 35 vertices are graceful**, which was verified by Fang in 2010 [12].

CHAPTER 4

VARIANTS OF GRACEFUL LABELING

In this chapter we discuss the variants of graceful labeling. These variants are obtained by modifications in graceful labeling. k-graceful labeling, Triangular graceful labeling, Odd triangular graceful labeling, Second order triangular graceful labeling, Fibonacci graceful labeling etc, are some of the variants of graceful labeling. Here we discuss about k-graceful labeling, triangular graceful labeling , odd triangular graceful labeling on some classes of graphs and an introduction to second order triangular graceful labeling, fibonacci graceful labeling.

k - Graceful Labeling

Definition 4.1: A k-graceful labeling of a graph G = (V, E) with q edges is an injection Ψ : V(G) to $\{0, 1, 2, ..., q + k - 1\}$ such that the corresponding edge labels is $\{k, k + 1, k + 2, ..., q + k - 1\}$.

Theorem 4.1: For all k $\in$ N, the cycle C_n, n $\equiv$ 0 (mod 4) is k-graceful [7].

Proof: Let C_n be a cycle with V(C_n) = $\{v_1, v_2, ..., v_{n-1}, v_n\}$. Consider the mapping Ψ: V(C_n) to $\{0, 1, 2, ..., n + k - 1\}$ defined by,

$$\Psi(v_i) = \begin{cases} \frac{i-1}{2} & ; i \ odd \\ n+k-\frac{i}{2} & ; i \ even \, and \, i \ \leq \frac{n}{2} \\ n+k-1-\frac{i}{2} & ; i \, even \, and \, i \ > \frac{n}{2} \end{cases}$$

and the mapping Ψ is injective. Also the induced mapping Ψ*: E(C_n) to $\{k, k+1, k+2, ..., n+k-1\}$ is bijective, given as Ψ*(u,v) = |Ψ(u) - Ψ(v)| for every (u,v) ∈ E(C_n) and u,v ∈ V(C_n), Thus Ψ is k-graceful labeling of C_n [7]. ⋄

Theorem 4.2: Paths P_n are k-graceful for all k [7].

Triangular Graceful Labeling

Definition 4.2: A graph G = (V,E) of order p and size q is called triangular graceful if there is an injection Ψ: V(G) to $\{0, 1, 2, ..., T_q\}$ where $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, T_n = (n(n+1))/2 such that the induced function on E(G) given by Ψ*: E(G) to $\{T_1, T_2, T_3, ...T_q\}$ defined as Ψ*(V_iV_j) = |Ψ(V_i)-Ψ(V_j)| for every V_iV_j ∈ E(G) is injective. The function Ψ is called triangular graceful labeling.

Theorem 4.3: The path P_m is triangular graceful, ∀ $m \geq 2$ [3].

Proof: Let $\{V_1, V_2, ..., V_m\}$ be the vertex set of P_m. Define a function

Ψ: V(P_m) to $\{0, 1, 2, ..., T_{m-1}\}$ as follows:

$\Psi(V_1) = 0$

$\Psi(V_2) = T_n$; n = m - 1

$\Psi(V_{2i}) = \Psi(V_{2i-2})$ - [m - (2i - 2)] ; i= 2, 3,...,[m/2]

$\Psi(V_{2i-1}) = \Psi(V_{2i-3})$ + (m - i) + (3 - i) ; i= 2, 3,...,[(m+1)/2]

and the edge labeling is given by $\Psi^*(V_iV_j) = |\ \Psi(V_i)$ - $\Psi(V_j)\ |$ for every $V_iV_j \in$ E(P_m) of the form $\{T_1, T_2, T_3, ...T_{m-1}\}$. Then Ψ and Ψ^* are 1-1. Hence the path P_m is triangular graceful $\forall$ m ≥ 2 [3]. $\diamond$

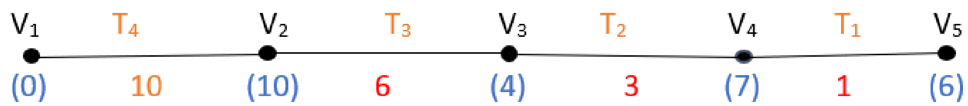


Figure 4.1 : Triangular graceful labeling of P_5

Theorem 4.4: The cycle C_n are triangular graceful, for n $\equiv$ 0 (mod 4) [3].

Proof: Let $V_1, V_2, ..., V_n$ be the vertices of C_n. Label the vertices as:

$\Psi(V_1) = 0$

$\Psi(V_n) = T_n$

$\Psi(V_2) = T_{n-1}$

$\Psi(V_{i-1}) = \Psi(V_i) \pm T_{i-2}$; i = n, n - 1,...,4

such that $|\ \Psi(V_{i-1})$ - $\Psi(V_i)\ | = T_{i-2}$ and $|\ \Psi(V_2)$ - $\Psi(V_3)\ | = T_1$

Then the set of edge values are $T_1, T_2, T_3, ...T_n$.

Thus C_n is triangular graceful for n $\equiv$ 0 (mod 4). $\diamond$

Theorem 4.5: The complete graph K_n is not triangular graceful, $\forall\, n \geq 3$ [3].

Proof: Consider K_3. To obtain a triangular graceful labeling, V(K_3) $\in$ {0,1,2,..,T$_3$} $= \{0, 1, 2, ..6\}$ so that E(K_3) $= \{T_1, T_2, T_3\} = \{1, 3, 6\}$. Assume that the vertex V_1 is labeled 0. The favourable outcomes for V_2 are 6, 1, 3. If $\Psi(V_2) = 6$. Then | $\Psi(V_1)$ - $\Psi(V_2)$ | $= 6 = T_3$. But there is no number $\alpha \in \{0, 1, .., 6\}$ such that | $\Psi(V_2)$ - α | $= 3$ or 1. Therefore K_3 is not a triangular graceful graph. Therefore K_n is not triangular graceful as it contains K_3. $\diamond$

Theorem 4.6: The Wheel W_n are not triangular graceful [3].

Proof: In wheel graph, we know that the central vertex is adjacent to all the other vertices. so we cannot label 0 to the central vertex. If this is the case then the labeling of the remaining vertex must be triangular numbers which will not give our required edge labeling. Moreover, the central vertex must be adjacent to 0, which makes the central value a triangular number, then the edge values will not again be triangular numbers. Thus wheels are not triangular graceful [3]. $\diamond$

Theorem 4.7: The complete bipartite graph $K_{m,n}$ is not triangular graceful, $\forall$ m, n ≥ 2 [3].

Proof: Let $\{V_1, V_2, ..., V_m\}$ and $\{U_1, U_2, ...U_n\}$ be two disjoint vertex set of $K_{m,n}$. Let $\Psi(U_1) = 0$. Then $V_1, V_2, V_3, ...V_m$ must be triangular numbers. But it is not possible to find a number α to label the vertices $U_2, U_3, ..., U_n$ to get | $\Psi(V_i)$ - α | a triangular number. Therefore $K_{m,n}$ is not a triangular graceful graph [3]. $\diamond$

Odd Triangular Graceful Labeling

Definition 4.3: A graph G = (V,E) of order p and size q is called an odd triangular graceful if there is an injection Ψ: V(G) to $\{0, 1, 2, ..., T_{2q-1}\}$ where T_i is the i^{th} triangular number such that the induced function on E(G) given by Ψ^*: E(G) to $\{T_1, T_3, T_5, ...T_{2q-1}\}$ defined as $\Psi^*(V_iV_j)$ = $|\Psi(V_i)$-$\Psi(V_j)|$ for every $V_iV_j \in$ E(G) is injective. The function Ψ is called an odd triangular graceful labeling [6].

Theorem 4.8: All paths P_n are odd triangular graceful graphs [6].

Proof: Let V(P_n) = $\{V_1, V_2, ..., V_n\}$. Then the size of P_n is q = n-1 and let T_q be the q^{th} triangular number.
Consider the mapping Ψ: V(G) to $\{0, 1, 2, ..., T_{2q-1}\}$, label the vertices as follows:

$\Psi(V_1) = T_{2q-1}$, $\Psi(V_2) = 0$, $\Psi(V_3) = 1$,
$\Psi(V_{i+3}) = \Psi(V_{i+2}) + (-1)^{i+1}\ T_{2q-(2i+1)}$, 1 ≤ i ≤ q - 2
$\Psi(V_n) = \Psi(V_{n-1}) + (-1)^n\ T_3$
which will give Ψ as an injective function and the induced function Ψ* on E(P_n) is given by Ψ*(u,v) = | Ψ(u) - Ψ(v) | ∀ u,v ∈ E(P_n). So Ψ is an odd triangular graceful labeling.Thus paths are odd triangular graceful graphs [6]. ◇

Theorem 4.9: All stars $K_{1,n}$ are odd triangular graceful graphs [6].

Proof: Let V($K_{1,n}$) =$\{V_0, V_1, ..., V_n\}$ and E($K_{1,n}$) = $\{V_0V_i/1 \leq i \leq n\}$. Then its order = n+1 and size = n.
Consider the mapping Ψ: V($K_{1,n}$) to $\{0, 1, 2, ..., T_{2q-1}\}$, label the vertices as follows:
$\Psi(V_0) = 0, \quad \Psi(V_i) = T_{2i-1}$; 1≤ $i \leq n-1$
$\Psi(V_n) = T_{2n-1} = T_{2q-1}$
Then the mapping Ψ is injective and the induced mapping Ψ* on E($K_{1,n}$) is given by Ψ*(u,v) = |Ψ(u) - Ψ(v)|, ∀ u,v ∈ E($K_{1,n}$) = $\{T_1, T_3, ...T_{2q-1}\}$. So Ψ is an odd triangular graceful labeling. Thus stars $K_{1,n}$ are odd triangular graceful graphs [6]. ◇

Graceful graphs need not be triangular graceful or odd triangular graceful. For example, complete graph K_3 is graceful. But it is neither triangular graceful or odd triangular graceful. Also some graphs are graceful, triangular graceful and odd triangular graceful. Example : stars $K_{1,n}$. Moreover, cycle C_6 is triangular graceful but not odd triangular graceful.

Definition 4.4: The minimum number of vertices whose removel from the graph G which makes the resulting graph odd triangular graceful is called the **odd triangular graceful number** denoted by O_g^tn(G). For exmaple, Cycles are not odd triangular graceful. But, if we remove one vertex from cycle, the resulting graph is a path which is odd triangular graceful. Therefore O_g^tn(C_n) = 1

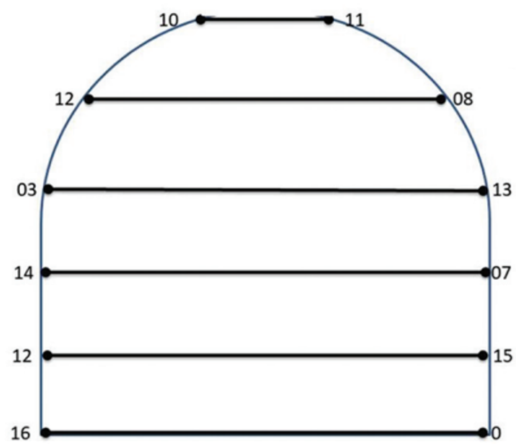
Definition 4.5: Let G be a graph with order p and size q. A **second order triangular graceful labeling** of graph G is an injection Ψ: V(G) to $\{0, 1, 2, .., S_q\}$ where S_q is q^{th} second order triangular number given by S_q = (q(q+1)(2q+1))/6 such that Ψ* is a bijection, Ψ*: E(G) to $\{S_1, S_2, ..., S_q\}$ defined as $\Psi^*(V_iV_j) = |\Psi(V_i)$-$\Psi(V_j)| \; \forall \; V_iV_j \in$ E(G). The graph that admits second order triangular graceful labeling is called second order triangular graceful graph [10].

Definition 4.6: A **fibonacci graceful labeling** of a graph G is an injection Ψ : V(G) to $\{0, 1, 2, ..., F_q\}$ where F_q is the q^{th} fibonacci number. Then the induced edge labeling Ψ*(u,v) = |Ψ(u) - Ψ(v)|, ∀ u,v ∈ E(G) is a bijection onto the set $\{F_1, F_2, ..., F_q\}$. The graph Gthat admits fibonacci graceful labeling is called fibonacci graceful graph [11].

CHAPTER 5

OTHER APPLICATIONS OF GRACEFUL GRAPHS

Application of graceful graphs in dental arch

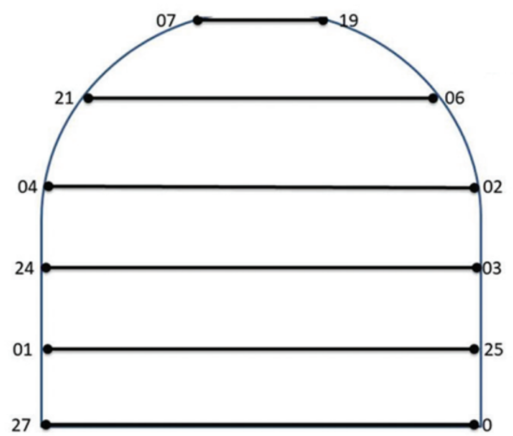


The dental arch can be divided into lower arch and upper arch. Right and left central incisors, lateral incisors, canines, first and second premolars, and molars make up each arch. Here, it's taken into account up until the first molars. On each side of the dental arch, there are six teeth overall, for a total of 12 teeth.

Each tooth in the arch is regarded as a vertex, and a line connecting the neighboring teeth and teeth of the same kind on the left and right side creates the edges. Apply graceful labeling to the graph where the vertex set and edge set is given by V(G) = $\{0, 1, 2, ..., 16\}$ and E(G) = $\{1, 2, ..., 16\}$.

Vertex labels and edge labels are discovered to be distinct during

labeling. In addition, the arrangement of the vertex labels follows a specific pattern. Utilizing graph labeling, one can evaluate how different teeth relate to the arch [5].
The dental arch can also be represented by k-graceful labeling.



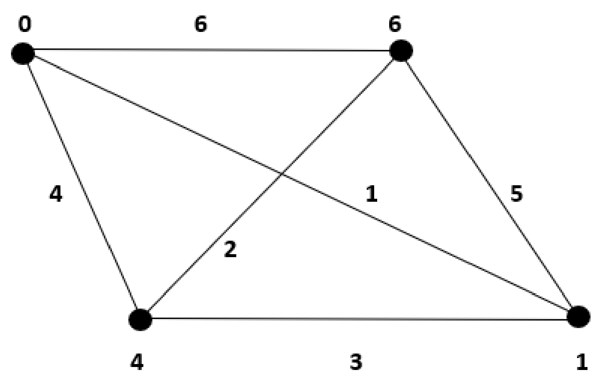
In k-graceful labeling of dental arch, let q = no of edges = 16, k= 12. Then the vertex set consist of labels from 0 to q+k-1 = 16+12-1 = 27 and the edge set consists of k, k+1,...., q+k-1.i.e V = $\{0, 1, 2, ..., 27\}$ and E = $\{12, 13, ..., 27\}$. Thus the dental arch can be represented by k-graceful labeling.
The dental arch can also be represented by odd graceful labeling in which vertex set V = $\{0, 1, ..., 2q - 1\}$ = $\{0, 1, 2..., 31\}$ and edge set consists of all odd labels <31. That is E = $\{1, 3, 5, ..., 31\}$.

Using graceful labeling the analysis of the arch's teeth might be done easily. Consequently, graceful labeling is an effective technique that enables the learning of complex patterns with ease and convenience in a variety of domains.

Coding Theory

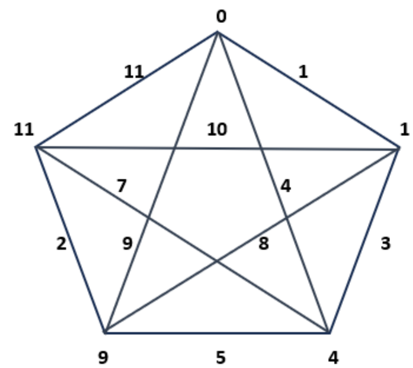
It is analogous to numbering a complete graph in a way that ensures that all edge numbers are distinct to create certain significant families of good non periodic codes for pulse radar and missile guidance. The time positions at which the pulses are transmitted are then determined by the vertex number. A full graph is one in which every pair of vertex is connected by an edge.

In K_4, p= |V| = 4, q = |E| = 6. Therefore Ψ: V to $\{0, 1, ...6\}$ and Ψ*: E to $\{1, 2..., 6\}$.



Yet K_5 is semi-graceful labeled. If the requirement that the edge lengths be consecutive is eased, this is referred to as a semi-graceful labeling. By including n+1 edge lengths to the graph, one edge length can be skipped.The semi-graceful labeling of K_5 is given below.

p = |V| = 5, q = |E| = 10. Therefore Ψ:V to $\{0, 1, 2, ..11\}$ and Ψ*: E to $\{1, 2, 3, 4, 5, 7, 8, 9, 10, 11\}$.



When edge length limitations are kept the same but vertex labels are permitted to extend beyond the largest edge length value, this is known as quasi-graceful labeling. This kind of elegant labeling makes it possible to extend the notion of coding [2].

Communication Networks

The lines connecting any two communication centers might be labeled with the difference between two center labels (i.e. vertex labels), if the communication network contained a fixed number n+1 of communication centers (i.e. vertex) and they were numbered 0, 1, .., n.

We would be able to mark the connections between each communication center if the grid were organized as an elegant graph, giving each connection a unique label. Such labeling has the benefit of allowing a straightforward algorithm to identify which two centers are no longer connected in the event that a link breaks [8].

CONCLUSION

There are still properties to be discovered about the graceful labeling of graphs, a subject of research for many years. From mathematicians to scientists who came in contact with it, graceful trees have been labeled as diseased. Numerous studies using graceful labels and graceful trees are still being conducted. This project presents some theoretical findings and provides a brief summary of the topic.

The issue is discussed in Chapter 2, along with several elegantly simple graph classes like cycles and wheels. We also provide required criteria for the existence of a graceful labeling for a graph. In Chapter 3, we concentrate on how to gracefully label trees, more precisely, how to approach the Graceful Tree Conjecture in several ways.

In chapter 4, we have discussed about different variants of graceful labeling.There are more variations of graph labeling. All these variations are found to tackle the graceful tree conjecture. Eventually in chapter 5, we have seen the widespread applications of graceful graphs in dental arch, coding theory and communication network.

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