MATRIX LIE GROUP

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MASTER'S DEGREE IN MATHEMATICS

By

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DECLARATION

I Nega Martin hereby declare that this project entitled **'MATIX LIE GROUP'** is a bonafide record of work done by me under the guidance of Toby Antony, Associate Professor, Department of Mathematics, Bharata Mata College, Thrikkakara and this work has not previously formed by the basis for the award of any academic qualification, fellowship or other similar title of any other University or Board.

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CERTIFICATE

This is to certify that the project entitled **MATRIX LIE GROUP** submitted for the partial fulfilment requirement of Master's Degree in Mathematics is the original work done by Nega Martin during the period of the study in the Department of Mathematics, Bharata Mata College, Thrikkakara under my guidance and has not been included in any other project submitted previously for the award of any degree.

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CONTENTS

INTRODUCTION

A group that is also a differentiable manifold is known as a lie group in mathematics. While groups define the abstracts, general concepts of multiplication and taking inverses, a manifold is a space that is similar to Euclidean space locally. Combining these two concepts creates a continuous group that allows for the multiplication and taking of the inverse of points.A lie group is produced if the multiplication and taking of inverses are further defined to be smooth (differentiable).

The idea of continuous symmetry can be naturally represented by Lie groups, one of whose well-known applications is the rational symmetry in three dimensions. In numerous areas of modern mathematics and physics, Lie groups are extensively used. Lie groups where first found by studying matrix sub groups G contained in GLn (R) or GLn(C), the groups of n $\times$ n invertible matrices over R or C.

Lie algebras are closely related to Lie group, which are groups that are also a smooth manifold, with the property that the group operations of multiplication and inversion are smooth maps. Any Lie group give rise to a Lie algebra.In contrast, there is a connected Lie group that is specific to covering for any one-dimensional Lie algebra over real or complex numbers. This correspondence between Lie groups and Lie algebra allows one to study Lie groups in terms of Lie algebras. Lie algebras and their representations are used extensively in physics, notably in quantum mechanics and particle physics.

In the first chapter we deal with some preliminaries of Lie groups, especially with Manifolds. The second chapter deals with Matrix lie groups and the relative concepts, where in the next chapter we discuss about Lie algebra and some related topics.In the fourth chapter we discuss about applications of matrix lie group.

Chapter 1

PRELIMINARIES

1.1 MANIFOLDS

Basic Definitions

1.1.1 Definition

A topological manifold $\boldsymbol{M}$ of dimension n is a topological space that is locally homoeomorphic to $\mathbb{R}^n$. This implies that for each point m in $\boldsymbol{M}$, there is neighbourhood U of m as well as a one-to-one correspondence map ϕ of U into some open set ϕ(U) in $\mathbb{R}^n$ such that the inverse map ϕ^{-1} : ϕ(U) $\rightarrow$ U is continuous. A manifold is a topological space that looks locally like a small piece of $\mathbb{R}^n$. The map ϕ is thought to define local coordinate functions X_1,X_2,....,X_n where each X_k is the continuous function from U into $\mathbb{R}$ given by X_k(m)=ϕ(m)$_k$[4]

1.1.2 Definition

A smooth manifold of dimension n consist of a topological manifold $\boldsymbol{M}$ and a distinct family of local coordinate system $(\mathrm{U}_\alpha,\phi_\alpha)$ with the following properties.[4]

1.Every point in $\boldsymbol{M}$ is contained in atleast one of the Uα's.
2.For any two of these coordinate systems $(\mathrm{U}_\alpha,\phi_\alpha)$, $(\mathrm{U}_\beta,\phi_\beta)$ the change of coordinates maps $\phi_\beta \circ {\phi_\alpha}^{-1}$ is a smooth map of the set $\phi_\alpha(\mathrm{U}_\alpha \cap \mathrm{U}_\beta) \subset \mathbb{R}^n$ onto the set $\phi_\beta(\mathrm{U}_\alpha \cap \mathrm{U}_\beta)$ $\subset \mathbb{R}^n$.[4]

This means that to create a smooth manifold, we begin with a topological manifold and then select a set of local coordinate systems that covers the entire manifold and such that at any time two coordinate systems are defined in overlapping regions, the expression for one set of coordinate system terms of the other is always smooth. Note that these coordinate systems must be chosen in order to provide smooth structure to the topological manifold $\boldsymbol{M}$. Some manifolds do not admit a smooth structure, when a smooth structure exists, it is not unique.[4]

1.1.3 Definition

A smooth local coordinate system to be any local coordinate system (U,ϕ). A function $f : \boldsymbol{M} \longrightarrow \mathbb{R}$ is called smooth if for each smooth local coordinate system (U,ϕ) , the function $f \circ \phi^{-1}$ is a smooth function on the set $\phi(\mathrm{U})$. In other words f is smooth if it is smooth in each smooth local coordinate system.

1.2 Tangent Space

1.2.1 Definition

The tangent space at m to $\boldsymbol{M}$,denoted $T_{m(\boldsymbol{M})}$, is the set of all linear map X from $C^{\infty}(\boldsymbol{M})$ into $\mathbb{R}$ satisfying[4]

1. The "product rule": X(fg)=X(f)g(m)+f(m)X(g) for all f and g in C^{∞}.[4]

2. Localization: If is equal to g in a neighbourhood of m, then X(f)=x(g).[4]

This is clearly a real vector space. A tangent vector at m is an element of T_m. If X_1, X_2,...,X_n is a local coordinate system, then each tangent vector X at m can be uniquely written as[4]

$$X(f)=\sum_{k=1}^{n} a_k \frac{\partial f}{\partial x_k}(m)$$

for some real constants a_1,a_2,...a_n. This indicates that if $\boldsymbol{M}$ is a manifold of dimension n, then $T_m(\boldsymbol{M})$ is a real vector space of dimension n for each m in $\boldsymbol{M}$.[4]

1.3 Submanifold of a vector space

1.3.1 Definition

If V is a finite dimensional real vector space, we can use a single, globally specified linear coordinate system to transform V into a smooth manifold.[4]
Given two vectors u, v in V, we can define the directional derivative of a given function at the point u in the direction of the vector v as

$$(D_v f)(u)=\frac{d}{dt}f(u+tu)|t=0$$

A smooth embedded sub manifold of dimension k if given any m_0 in $\boldsymbol{M}$, there exist a smooth coordinate system (ϕ,U) defined in a neighbourhood U of m_0 such that for any m $\in$ U, m is in U$\cap$M if and only if ϕ(m) is in $\mathbb{R}^k \subset \mathbb{R}^n$.[7]

1.3.2 Proposition

Let $\boldsymbol{M}$ be a smooth embedded sub manifold of a finite dimensional real vector space V. The tangent space to $\boldsymbol{M}$ at m is the set of all u in V such that there exists a smooth curve y in M with y(0)=m and $\frac{dy}{dt}$=u, for any m $\in$ M.

1.3.3 Definition

A complex manifold is a smooth manifold of dimension 2n in which the basic coordinate patches (U_α,ϕ_α) have the feature that change of coordinates maps $\phi_\beta \circ {\phi_\alpha}^{-1}$ is holomorphic for each α and β. Here $\mathbb{R}^{2n}$ is identified with $\mathbb{C}^n$ and holomorphic means the same as complex analytic.[4]
If V is a complex vector space, the subset $\boldsymbol{M}$ of V is known as an embedded complex sub manifold of dimension k if, for each $m_\circ \in \boldsymbol{M}$, there exist a holomorphic local coordinate system (ϕ,U) defined in neighbourhood U of $m_\circ$ such that for any m $\in$ U, m is in U$\cap\boldsymbol{M}$ if and only if ϕ(m) is in $\mathbb{C}^k \subset \mathbb{C}^n$.[4]

Some examples of manifolds:

One – dimensional manifolds includes lines and circles. Two dimensional manifolds are also called surfaces. Examples includes the plane, the sphere, and the torus, which can all be embedded in three-dimensional real space, but also the Klein bottle and real projective plane, which will always self-intersect when immersed in three-dimensional real space. Although a manifold locally resembles Euclidean space, meaning that every point has a neighbourhood homoeomorphic to an open subset of Euclidean plane, because it has the global topological property of compactness that Euclidean space lacks, but in a region, it can be charted by means of map projections of the region into the Euclidean plane. Manifolds need note closed; thus, a line segment without its end points is a manifold.

And they never countable, unless the dimension of the manifold is 0. Putting these freedoms together, other examples of manifolds are a parabola, a hyperbola, and the locus of points on a cubic curve $y^2 = x^3 - x$.

1.4 GROUPS

Basic Definitions

1.4.1 Definition

A group G is a non-empty set together with a binary operation $*$ satisfying the following properties;

1) Closure property: $a * b \in G$ for all $a,b \in G$

2) Associative law: $a * (b * c) = (a * b) * c$ for all $a,b,c \in G$

3) Identity element: For all $a \in G$ $\exists$ $e \in G$ such that $(a * e)=(e * a) = a$

4) Inverse element: For all $a \in G$ $\exists$ $a^{-1} \in G$ such that $(a * a^{-1}) = (a^{-1} * a) = e$

1.4.2 Examples

1) $\mathbb{Z}$, the set of integers, under the operation addition.

2) $\mathbb{R}$, the set all real numbers, under the operation addition.

3) $\mathbb{R}^*$, the set of non-zero real numbers, under the operation multiplication.

1.4.3 Proposition(Basic group properties)

For any group G

- The identity element of G is unique[10].
- For each $a \in G$, the inverse a^{-1} is unique[10].
- For any $a \in G$, $(a^{-1)})^{-1} = a$.
- For any $a,b \in G$, $(ab^{-1}) = b^{-1}a^{-1}$[10].
- For any $a,b \in G$,the equation $ax = b$ and $ya = b$ have distinct solutions or in other words the left anf right cancellation laws apply[10].

1.4.4 Definition

The general linear group, denoted $\mathbf{GL}(n,\mathbb{R})$, consists of the set of invertible $n \times n$ matrices. We know that multiplication of invertible $n \times n$ matrices is associative, and each invertible matrix has an inverse and an identity, $\mathbf{GL}(n,\mathbb{R})$, which forms group when multiplied.[10]

1.5 Basic definitions of Lie Groups

1.5.1 Definition

A (real) Lie group is a set G with two structures: group and manifold. These structures are compatible in the following sense: multiplication map $G \times G \longrightarrow G$ and inversion map $G \longrightarrow G$ are smooth maps. A Lie group morphism is a smooth map that maintains the group operation: f(gh)=f(g)f(h),f(1)=1. For image and kernel of a morphism, we shall use the standard notation Imf,Kerf.[7]

1.5.2 Definition

A complex Lie group is a set G with two structures: group and complex analytic manifold. These structures are compatible in the following sense: multiplication map $G \times G \longrightarrow G$ and inversion map $G \longrightarrow G$ are analytic maps. A complex Lie group morphism is an analytic map that maintains the group operation: f(gh)=f(g)f(h), f(1)=1.[7]

$\mathbb{R}^n$ with the group operation given by addition , All usual groups of linear algebra,such as GL(n,$\mathbb{R}$) ,SL(n,$\mathbb{R}$) are examples of Lie groups.

1.5.3 Definition

A closed Lie subgroup H of a (real or complex) Lie group G is a subgroup that also happens to be a submanifold (for comples Lie groups,it will be a complex manifold).[7]

Chapter 2

MATRIX LIE GROUPS

2.1 Definitions of a Matrix Lie Group

The general linear group over the real numbers **GL**(n,$\mathbb{R}$)(or complex numbers **GL**(n,$\mathbb{C}$)) is the collection of all n×n invertible matrices having real entries(or complex entries). Let $\mathbb{M}_n(\mathbb{C})$ be the space containing all matrices with complex entries.

Let A_m denote a series of complex matrices in $\mathbb{M}_n(\mathbb{C})$. A_m is said to converge to a matrix A if each entry in A_m converges (as m $\longrightarrow \infty$) to the corresponding entry of A. A matrix Lie group is any subgroup G of **GL**(n,$\mathbb{C}$) that has the property: If A_m is any matrix sequence in G, and A_m converges to some matrix A then either A$\in$G or A is not invertible.[8]

This is equivalent to saying that a matrix Lie group is a closed subgroup of **GL**(n,$\mathbb{C}$).[1]

2.2 Examples of Matrix Lie Groups

2.2.1 The general linear groups GL(n,$\mathbb{R}$) and GL(n,$\mathbb{C}$)

The general linear groups (over $\mathbb{C}$ or $\mathbb{R}$) are matrix Lie groups in and of themselves. **GL**(n,$\mathbb{C}$) is , of course, a subgroup of itself.[3]
If A_m is a sequence of matrices in **GL**(n,$\mathbb{C}$) and A_m converges to A, then by the definition of **GL**(n,$\mathbb{C}$) ,either A $\in$ **GL**(n,$\mathbb{C}$) or A is not invertible.**GL**(n,$\mathbb{R}$) is subgroup of **GL**(n,$\mathbb{C}$), hence by definition of matrix Lie group **GL**(n,$\mathbb{R}$) is matrix Lie group.[1]

2.2.2 The special Lie groups SL(n,$\mathbb{R}$) and SL(n,$\mathbb{C}$)

The special linear group (over $\mathbb{R}$ or $\mathbb{C}$) consist of all n$\times$n invertible matrices with determinant one. Both of these are subgroups of **GL**(n,$\mathbb{C}$). Because the determinant is a continuous function, A_n is a sequence of matrices with determinant one. As a result **SL**(n,$\mathbb{R}$) and **SL**(n,$\mathbb{C}$) are matrix Lie groups.[3]

2.2.3 The orthogonal and special orthogonal groups, O(n) and SO(n)

If the column vectors that make up A are orthonormal where A is an n$\times$n real matrix, then A is orthogonal, that is, if $\sum_{i=1}^{n} A_{ij}A_{ik} = \delta_{jk}$ $1 \leq j, k \leq n$ where δ_{jk} is the Kronecker delta.[3]
Equivalently, If A retains the inner product, it is orthogonal, namely if $\langle x, y\rangle = \langle A_x, A_y\rangle$ for all vectors x,y in $\mathbb{R}^n$. Another equivalent definition is that A is orthogonal if $A^{tr}A$ = I (where A^{tr} is the transpose of A)[4]
The orthogonal group **O**(n) is the set of all n$\times$n real orthogonal matrices and is a sub-

group of **GL**(n,$\mathbb{C}$). Because the relation A^{tr}A=I is retained under taking limits, the limit of sequence orthogonal matrices is orthogonal. As a result **O**(n) is a matrix Lie group.[4]

If A is an orthogonal matrix, then det A=±1.

The special orthogonal group **SO**(n) is the set of all n×n matrices with determinant one. This is clearly a subgroup of **O**(n), and so **GL**(n,$\mathbb{C}$). Furthermore both orthogonality and the property of having determinant are preserved under limits, therefore **SO**(n) is a matrix Lie group.[3]

2.2.4 The unitary and special unitary group

If the column vectors of A are normal, where A is an n×n complex matrix then A is said to unitary,that is,[3]

$$\sum_{i=1}^{n} \overline{A}_{ij}\mathrm{A}_{ik} = \delta_{jk}$$

Equivalently if A retains the inner product then A is unitary, namely if $\langle x, y\rangle = \langle A_x$,$\mathrm{A}_y\rangle$ for all x,y in $\mathbb{C}^n$. In other words A is unitary if $A^*A = I$ where A^* is the adjoint of A. For all unitary matrices |det A| = 1.[3]

The unitary group U(n) is the set of all n×n unitary matrices. This is clearly a subgroup of GL(n,$\mathbb{C}$). Since the limit of unitary matrices is unitary, this implies that U(n) is a matrix Lie group. The special unitary matrix SU(n) is the set of unitary matrices with determinant one.[2]

2.2.5 The groups $\mathbb{R}^*$, $\mathbb{C}*$, $\mathbf{S}^1$ and $\mathbb{R}^n$

Under multiplication, the group $\mathbb{R}^*$ of non-zero real numbers is isomorphic to $\mathbf{GL}(1,\mathbb{R})$. As a result, we will refer to $\mathbb{R}^n$ as matrix Lie groups. Similarly, the group $\mathbb{C}^*$ of non-zero complex numbers under multiplication is isomorphic to $\mathbf{GL}(1,\mathbb{C})$ and the group S^1 of complex numbers with absolute value one is isomorphic to $\mathbf{U}(1)$.[3]

2.3 Compactness

2.3.1 Definition

If the following two conditions met, then a matrix Lie group G is said to be compact to a matrix[3]

a. If A_m is any sequence of matrices in G and converge to a matrix A then A is in G.[3]

b. There exist a constant C such that for all $A \in G$, $|A_{ij}| \leq C$ for all $1 \leq i,j \leq n$.[3]

2.3.2 Examples of compact groups

The groups O(n) and SO(n) are compact. Because the limit of orthogonal matrices is orthogonal and the limit of matrices with determinant one has determinant one, property (a) is satisfied. Because A is orthogonal, the column vectors of A have norm one, hence property (b) is satisfied, and hence $|A_{kl}| \leq 1$ for all $1 \leq k,l \leq n$. By a similar argument we can show that U(n) and SU(n) are compact.[4]

2.3.3 Examples of non-compact groups

All other examples of matrix Lie groups are non-compact. The groups **GL**(n,$\mathbb{R}$) and **GL**(n,$\mathbb{C}$) violate the property (a). The groups **SL**(n,$\mathbb{R}$) and **SL**(n,$\mathbb{C}$) violate (b).

2.4 Connectedness

2.4.1 Definition

A matrix Lie group G is said to be connected if a continuous path A(t) exists between any two matrices A and B in G, a $\leq$ t $\leq$ b, lying in G with A(a)=A and A(b)=B.
A matrix Lie group G which is not connected can be decomposed (uniquely) as a union of many sections, called components, such that two members of the same component can be united by a path, but two elements of different components cannot be joined.[4]

2.4.2 Proposition

If G is a matrix Lie group, then the component of G containing the identity element is a subgroup G.[4]
Proof:
Let A and B be matrices in the component containing the identity element. This means that continuous pathways A(t) and B(t) exist with A(0)=B(0)=1, A(1)=A, and B(1)=B. Then A(t)B(t) is a continuous path that begins with I and ends with AB. Thus, the product of two identity component parts is again in the identity component. Furthermore because $A(t)^{-1}$ is a continuous map beginning at I, the inverse of any element of the identity component is also in the identity component.Thus the identity component

is a subgroup.[4]

2.4.3 Proposition

The group $\mathbf{GL}(n,\mathbb{R})$ is not connected, but has two components. These are $\mathbf{GL}(n,\mathbb{R})^+$, the set of n×n real matrices with positive determinant, and $\mathbf{GL}(n,\mathbb{R})^-$ the set of n×n real matrices with negative determinant.[4]

Proof:

If det A > 0 and det B < 0, any continuous path connecting A and B must include a matrix with determinant zero and hence pass outside of $\mathbf{GL}(n,\mathbb{R})$. As a result $\mathbf{GL}(n,\mathbb{R})$ is not connected.[4]

Both $\mathbf{GL}(n,\mathbb{R})^+$ and $\mathbf{GL}(n,\mathbb{R})^-$ are connected. Assume C is any matrix with a negative determinant, and A and B are in $\mathbf{GL}(n,\mathbb{R})^-$. Then $C^{-1}A$ and $C^{-1}B$ are in path joining A and B in $\mathbf{GL}(n,\mathbb{R})^-$.[4]

2.4.4 Theorem

Every matrix Lie group is a smooth embedded submanifold of $\mathbb{M}_n(\mathbb{C})$ and thus a Lie group.

2.4.5 Definition

Consider G and H to be matrix Lie groups. A Lie group homomorphism is a map ϕ from G to H, if[4]

1. ϕ is a group homomorphism[4]
2. ϕ is continuous.

It is called a Lie group isomorphism if in addition ϕ is one-to-one and onto and the inverse map ϕ^{-1} is continuous.[4]

2.4.6 Theorem

Let G and H be Lie group and ϕ be the group homomorphism from G to H. If ϕ is continuous, it is also smooth.[4]

Chapter 3

MATRIX LIE GROUPS AND THEIR LIE ALGEBRAS

3.1 Algebra

3.1.1 Definition

An algebra A over K is a vector space over K together with a bilinear map A×A $\longrightarrow$ A denoted (x,y) $\longrightarrow$ xy. In symbols we have:[9]

- x(y+z)=xy+xz and (x+y)z=xz+yz for all (x,y,z) $\in$ A^3[9]
- (ax)(by)=(ab)(xy) for all (a,b) $\in$ K^2,(x,y) $\in$ A^2[9]

3.1.2 Proposition

Let A be an algebra, and the vector space A plus the multiplication defined by (x,y) $\longrightarrow$ yx is another algebra called the opposite algebra and indicated by A^{op}.[9]

3.1.3 Definition

An endomorphism D of an Algebra A is called a derivation of A if the equivalence D(x,y)= D(x)+D(y) holds for every (x,y) $\in$ A^2[9]

3.1.4 Proposition

The kernel of a derivation is a subalgebra of A[9]

Proof:

Let the kernel contain x and y.The presence of xy in the kernel must be demonstrated. This follows from what a derivation is defined as.[9]

3.1.5 Proposition

If D_1 and D_2 are derivations of an algebra A, then the commutator $[D_1,D_2]=D_1D_2$-D_2D_1 is a derivation of A as well.[9]

3.2 The notion of Lie algebra

3.2.1 Definition

If the following axioms are satisfied a vector space L over a field F, with an operation L×L ,denoted (x,y) $\longrightarrow$ [x,y] and known as the bracket or commutator x and y is termed a Lie algebra over F:[11]

L_1:The bracket operation bilinear

$[ax + by, z] = a[x, z] + b[y, z]$

$[z, ax + by] = a[z, x] + b[z, y] for all x, y, z \in$ L

L_2:[x,x]=0 $\forall$ x$\in$L

L_3:[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0 x,y,z$\in$L

Axiom L_3 is called the Jacobi identity.

3.2.2 Result

In a lie group L,[x,y]=-[y,x]

Proof:

[x+y,x+y]=[x,x+y]+[y,x+y]

=[x,x]+[x,y]+[y,x]+[y,y]

By L_2 [x+y,x+y]=0

That is;

[x,x]+[x,y]+[y,x]+[y,y]=0 and [x,x]=0 [y,y]=0

Therefore,

[x,y]+[y,x]=0

[x,y]=-[y,x][6]

3.2.3 Result

In a Lie algebra L, if char F $\neq$ 2 then [x,y]=-[y,x] $\forall$ x,y $\in$ L implies that [x,x]=0

Proof:

Given char F $\neq$ 2. Also given that [x,y]=-[y,x]

put x=y then, [x,x]=-[x,x]

i.e,

2[x,x]=0

i.e,

$$[x,x]=0$$

3.2.4 Definition

A subspace K of L is called a subalgebra if $[x,y] \in K \; \forall \; x,y \in K$.

3.2.5 Definition

Two Lie algebra L and L' are said to be isomorphic if there exit a vector space isomorphic $\phi : L \longrightarrow L'$ fulfilling $\phi([x,y])=[\phi(x)\phi(y)] \; \forall \; x,y \in L$ and then ϕ is said to be an isomorphism of Lie algebra.[6]

3.2.6 Proposition

Any sub algebra of a Lie algebra, any quotient algebra of a Lie algebra, any product algebra of a Lie algebras is again a Lie algebra.

3.2.7 Proposition

The opposite algebra A^{op} of Lie algebra A is a Lie algebra once more and the morphism $A^{op} \longrightarrow A$ is an isomorphism which is defined by $x \longrightarrow x$.[9]

3.3 Examples of Lie algebra

3.3.1 *gl(V)*

If V is a finite dimensional vector space over F, denote by **E**nd V The set of linear transformations V $\longrightarrow$ V. As a vector space over F, **E**nd V has dimension n^2,[11]
and define a new operation,[x,y]= xy-yx, called the bracket of x and y. With this operation **E**nd V become a Lie algebra and it is the general linear algebra ***gl(V)***.[6]
For,

[ax+by,z]=[ax+by]z-z[ax+by]

=axz+byz-azx-bzy

=a[x,z]+b[y,z]

[ax+by,z]=a[x,z]+b[z,y]

z[ax+by]=azx+bzy-axz-byz

=a[z,x]+b[z,y]

[z,ax+by]=a[z,x]+b[z,y]

Therefore L_1 satisfied

[x,x]=xx-xx=0

Therefore L_2 satisfied

[x,[y,z]]+[y,[z,x]]+[z,[x,y]]

=[x,yz-zy]+[y,zx-xz]+[z,xy-yx]

=x(yz-zy)+y(zx-xz)+z(xy-yx)-(yz-zy)x-(zx-xz)y-(xy-yx)z

=xyz-xzy+yzx-yxz+zxy-zyx-yzx+zyx-zxy+xzy-xyz+yxz

=0

L_3 also satisfied.

Therefore ***gl**(V)* is a lie algebra.

Note:Any subalgebra of a Lie algebra ***gl**(V)* is called a linear algebra.

3.3.2 *sl(V)*

Let V be a vector space with finite dimension over a field F. The Lie algebra ***gl**(V)* is thus identified with a set of n×n matrices $\boldsymbol{gl}_n(F)$ where n is the dimension of V. The set of all matrices with trace zero $\boldsymbol{sl}_n(F)$ is a sub algebra of $\boldsymbol{gl}_n(F)$ and we denote it as ***sl**(V)*.

Proof:

$$\mathrm{tr}[x,y]=\mathrm{tr}(xy)-\mathrm{tr}(yx)$$

(because the matrix trace preserves bilinearity)[10]

tr([x,[y,z]]+[y,[z,x]]+[z,[x,y]])=tr[x,0]+tr[y,0]+tr[z,0]=0[10]

3.3.3 Example

The set of all anti symmetric matrices with the trace zero represented by $\mathbf{so}_n$ forms a Lie algebra with the commutator acting as the Lie bracket.

3.3.4 Example

Any vector space can be made into a Lie algebra with the trivial bracket:

$[v,w]=0 \ \forall \ v,w \in V$.

3.3.5 Example

We can show that the real vector space $\mathbb{R}^3$ is a Lie algebra. Let a,b,c represents arbitrary vectors in $\mathbb{R}$ and let α, β and γ be arbitrary scalars.

1. (a $\times$ b) = -(b $\times$ a)

2. a $\times$ (βb + γc) = β(a $\times$ b) + γ (a $\times$ c) and

   (αa + βb) $\times$ c = α(a $\times$ c) + β(b $\times$ c)

These are the properties of the cross product.

Now put a=b, (a $\times$ a)=-(a $\times$ a) $\rightarrow$ (a $\times$ a)=0 by 1

By the above properties cross product is skew symmetric and bilinear.

By vector triple product expansion x $\times$ (y $\times$ z) = y(x,z)-z(x,y)

To show that the cross product satisfies the Jacobi identity:

[x[y,z]]+[y[z,x]]+[z[x,y]] = x$\times$(y$\times$z)+y$\times$(z$\times$x)+z$\times$(x$\times$y)

=y(x.z)-z(x.y)+z(y.x)-x(y.z)+x(z.y)-y(z.x)

=0 (since dot product is commutative)

Therefore,by definition the real vector space $\mathbb{R}^3$ is a Lie algebra.[2]

3.4 Lie algebras of Derivations

3.4.1 Introduction

The derivative of f over g is a linear operator that obeys the Leibiniz rule.:

1) (fg)'=f'g+fg'

2) (αf)'=αf' where α is any scalar.[10]

3.4.2 Definition

The derivation of an algebra A over a field F is a linear map δ such that δ(f,g) = f(δg) + δ(f)g $\forall$ f,g $\in$ A. The set of all derivation of A is represented by Der(A). Given $\delta \in$ Der(A) f,g $\in$ A and $\alpha \in$ F.[10]

By property 2

$(\alpha\delta)$(fg) = $\alpha\delta(fg)$ =α(fδ(g)+δ(f)g) = αf$\delta(g)$+$\alpha\delta$(f)g[10]

Where F is a field , the Leibiniz rule is satisfied if and only if αf=fα.[10]

3.4.3 Example

Let x and y $\in$ End(v) and δ,δ' $\in$ Der(V).

By the definition of the commutator, [δ,δ'] = ($\delta\delta$'-δ'δ)

- Der(V) is a vector space of **E**nd(V):

  δ[x,y]=δ(xy-yx)[10]

  =xδ(y)+δ(x)y-(yδ(x)+δ(y)x)

  =δ(y)x-yδ(x)+xδ(y)-δ(y)x

  =[δ(x),y]+[x,δ(y)][10]

- The commutator δ,δ'-of two derivations δ,δ' $\in$ Der(V) is again a derivation.[10]

([δ,δ'](x))y+x([δ,δ'](y))=(($\delta\delta$'-δ'δ)(x))y+x($\delta\delta$'-δ'δ)(y)[10]

=($\delta\delta$'(x)-$\delta\delta$'(x))y+x($\delta\delta$'(y)-δ'δ(y))

=$\delta\delta$'(x)y-δ'δ(x)y+x$\delta\delta$'(y)-xδ'δ(y)

=δ(δ'(x)y)+δ'(x)δ(y)+δ(x)δ'(y)-xδ'δ(y)[10]

$$-\delta'(\delta(x)y)-\delta(x)\delta'(y)-\delta'(x)\delta(y)-x\delta(y)$$

$$=\delta(\delta'(xy))-\delta'(\delta(xy))$$

$$=[\delta,\delta'](xy)=\delta(\delta'(x)y+\delta'(y))-\delta'(\delta(x)y+x\delta(y))$$

3.4.4 Definition

Given x ∈ L, the map y ⟶ [x,y] is an endomorphism of L ,designated as $\mathbf{ad}_x$, where $\mathbf{ad}_x$ is an inner derivation. Derivation of the form [x,[y,z]]=[[x,y],z]+[y,[x,z]] are inner all other are outer.[10]

Note: The collection of all derivations, Der(V) satisfies skew symmetry. Der(V) satisfy the Jacobi identity according to the above definition. As a result Der(V) defines a Lie algebra.[10]

3.4.5 Definition

The adjoint representation of a Lie algebra is the map L ⟶ Der(V) sending x to $\mathbf{ad}_x$[10]

3.5 Ideals and Homomorphisms

3.5.1 Definition

A Lie algebra's (L) subspace I is said to be an ideal of L if x ∈ L together imply [x,y] ∈ I. All Lie algebra ideals are two-sided by definition due to skew symmetry. That is, if [x,y] ∈ I,then [y,x] ∈ I.[5]

The kernel of a Lie algebra L and L itself are trivial ideals contained in every Lie algebra.[5]

3.5.2 Example

The set of all inner derivations $\mathbf{ad}_x \in$ I, is an ideal of Der(L)

Let $\delta \in$ Der(V). By definition of inner derivations $\forall$ y $\in$ L[10]

$[\delta,\mathbf{ad}_x](y)=(\delta(\mathbf{ad}_x)-(\mathbf{ad})_x)\delta)(y)$

$=\delta[x,y]-\mathbf{ad}_x(\delta(y))$

$=[\delta(x),y]+[x,\delta(y)]-[\mathbf{ad}_x(\delta(y))]$

$=\mathbf{ad}_x(\delta(x)y)$

Therefore by above definition $\mathbf{ad}_x$ is an ideal of Der(V)[10]

3.5.3 Definition

The centre of a Lie algebra L is a set

$Z(L)=\{z\in L \mid [x,z]=0\ \forall\ x\in L\}$[10]

The collection of elements in L for which the adjoint action $\mathbf{ad}_x$ provides the zero derivation is represented by the center.[10]

3.5.4 Definition

The centralizer of a subset X of L is defined to be

$C_L(X)=\{x\in L \mid [x,X]=0\}$[10]

By Jacoby;$C_L(X)$ is a sub algebra of L where $C_L(L)=Z_L$

3.5.5 Definition

If the Lie algebra L has no ideals except itself and 0, if moreover[L,L] $\neq$ 0,we call L simple.

3.5.6 Result

Z(L) is an ideal

Proof:

First we prove that Z(L) is a subspace of L.

For let $z_1,z_2 \in L$ and $c \in L$

Then $[x,z_1]=0\ \forall\ x \in L$

$[x,z_2]=0\ \forall x \in L$

Consider $[x,cz_1+dz_2]=c[x,z_1]+d[x,z_2]=0$

$\therefore\ cz_1+dz_2 \in L$

$\therefore$ Z(L) is a subspace of L.

Next we have to show that $[x,y] \in Z(L)$ where $x \in L$, $y \in Z(L)$.

Let $x,z \in Z(L)$ and $y \in Z(L)$ be arbitrary

Now by Jacoby identity we get $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$

Thus we get $[z,[x,y]]=0\ \forall\ z \in L$

$\therefore\ [x,y]=0$

Z(L) is an ideal of L.

3.5.7 Definition

A Lie algebra L is called an Abelian Lie algebra if [x,y]=0 $\forall$ x,y $\in$ L.

3.5.8 Definition

ϕ: L $\longrightarrow$ L' is a linear transformation which is referred to as a homomorphism if

ϕ([x,y]) = [ϕ(x), ϕ(y)] $\forall$ x,y $\in$ L

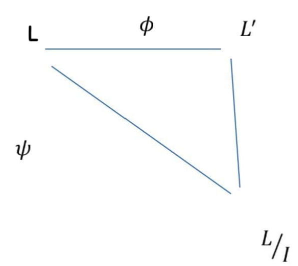
ϕ is said to be a monomorphism if Kernel ϕ=0 and an epimor if Imϕ = L'.

3.5.9 Proposition

a) If ϕ: L$\longrightarrow$ L' is a homomorphism of Lie algebras, then $\frac{L}{Kernel\phi} \cong$ Imϕ.[6]

If I is any ideal of L included Kernel ϕ, there exist a unique homomorphism

ψ: $\frac{L}{I} \longrightarrow$ L' making in the following diagram commute:[6]



b) If I and J are ideals of I such that I $\subset$ J, then $\frac{J}{I}$ is an ideal of $\frac{L}{I}$ and $\frac{(\frac{L}{I})}{(\frac{J}{I})}$ is naturally isomorphic to $\frac{L}{J}$.

c) If I and J are ideals of L, there is a natural isomorphism between $\frac{(I+J)}{J}$ and $\frac{I}{I\cap J}$.[6]

A representation of Lie algebra is a homomorphism ϕ: L $\longrightarrow$ $gl(V)$[6].

3.6 The Lie algebra of a Lie group

Let G be a matrix Lie group. We define Lie(G) by Lie(G)={A $\in$ gl(n): e$^{tA}\in$G, $\forall$ t $\in$ R}. Allow us to demonstrate that Lie(G) is a vector subspace of gl(n). Let A,B $\in$ Lie(G). Then we have e$^{t(A+B)}$ = lim k $\longrightarrow \infty$ (e$^{\frac{tA}{k}}$e$^{\frac{tB}{k}}$)^k $\in$ G, where the final conclusion makes advantage of the fact that G is enclosed in GL(n). As a result A+B $\in$ Lie(G). Also if A $\in$ Lie(G) and $\alpha \in$ R then definitely αA $\in$ Lie(G). As a result Lie(G) is also closed under scalar multiplication thus a vector subspace of gl(n).[1]

Following that, we demonstrate that Lie(G) is closed under the Lie bracket in [x,y] = x$\circ$y-y$\circ$x.[1]

That is, we show [A,B] $\in$ Lie(G) $\forall$ A and B in Lie(G) where [A,B]=AB-BA. First we require the following fundamental result.[4]

3.6.1 Lemma

Let G be a matrix Lie group. Let A $\in$ G and X $\in$ Lie(G) then AXA^{-1} $\in$ Lie(G).[4]

Proof:

Note that e$^{tAXA^{-1}}$ = Ae^{tx}A^{-1} $\in$ G for every t $\geq$ 0 and hence the result.

Now we get back at showing [A,B] $\in$ Lie(G) $\forall$ A,B $\in$ Lie(G).[4]

Let us define, Λ(t)=e^{tA}Be^{-tA}[1]

Λ(t) $\in$ Lie(G) $\forall$ t $\geq$ 0

Next we observe Λ'(t) = Ae^{tA}Be^{-tA}-e^{tA}BAe^{-tA}.

$\therefore$ AB-BA=Λ'(0)=lim h $\longrightarrow$ 0 $\frac{\Lambda(h)-B}{h}$ $\in$ Lie(G)[1]

where the last conclusion is derived from the previously established fact that Lie(G) is a vector subspace of gl(n) and thus closed in norm topology of gl(n).[1].

3.6.2 Theorem

Let G be a matrix Lie Group.Then Lie(G) is a Lie subalgebra of gl(n) with the Lie bracket[1].

Chapter 4

APPLICATIONS OF MATRIX LIE GROUPS

Matrix Lie groups have many applications in many fields such as physics, engineering, computer science and economics. Some of the most common uses of Matrix Lie groups include:

- Robotics: Matrix Lie groups are used to represent the orientation and position of a robot arm and calculate changes between them. This allows the robot to easily perform complex tasks such as material handling and assembly, with precision and accuracy.
- Computer Graphics: Matrix Lie groups are used in computer graphics to represent transformations of 3D objects such as rotation, translation, and scaling. This provides realistic simulation and animation of objects in a virtual environment.
- Quantum mechanics: Matrix Lie groups are used in quantum mechanics to represent the symmetries of physical systems. This allows calculation of important fac-

tors such as energy level, transition probability and break point.

- Control Theory: Matrix Lie groups are used in control theory to represent the dynamics of system with symmetry . This allows the design of control algorithms that can stabilize and control the behavior of complex systems.

- Econometrics: Matrix Lie groups are used in econometrics to model the evolution of an economy over time. This allows analysis of business transactions and prediction of future results.

- Differential Geometry: Matrix Lie groups are used in differential geometry to study the geometry of curved surfaces. This allows the development of mathematical tools that describe the behavior of bodies in curved space, such as the motion of planets around the sun.

In general, matrix Lie groups are widely used in many different fields and are important tools for understanding the behavior of complex systems.

In robotics, matrix Lie groups are used to represent the orientation and position of a robot arm and to calculate changes between them.

The most commonly used Lie groups in robotics are the special orthogonal group SO(3) and the special Euclidean group SE(3).

The special orthogonal group SO(3) is used to represent the rotation of a rigid body in three-dimensional space. This group consists of all the 3x3 orthogonal matrices with determinant equal to 1. The Lie algebra of SO(3) is the set of all 3x3 skew-symmetric matrices, which represent infinitesimal rotations. The Lie bracket operation in SO(3) corresponds to the cross product of vectors, which represents the effect of successive infinitesimal rotations.

The special Euclidean group SE(3) is used to represent the rigid body of the robot in three-dimensional space, including rotation and translation. This group includes all 4x4 matrices holding distances and angles. The Lie algebra of SE(3) is the set of all 4x4 matrices that are obliquely symmetric about the block diagonal. The separator operation in SE(3) corresponds to the cross product of vectors and the inner product of a 3x3 submatrix vector.

Robotic operators can represent and control robot operations in a rigorous mathematical fashion using matrix Lie groups; this allows the development of efficient algorithms for planning, execution for improvement and management. For example, the use of Lie groups has allowed the development of interpolation algorithms to ensure the smoothness and uniformity of a robot arm. In addition, the Lie group method can be used to predict the result of the robot from measurement, which is an important task in many applications such as product management and assembly.

In robotics, the motion of a robot can be expressed as a series of transformations in which each transformation corresponds to a change in position and direction. These transformations can be represented by matrices, and the group of all matrices representing efficient transformations is called the special Euclidean group SE(3). The elements of this group are 4x4 matrices of the following format:

$$\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

where R is a 3×3 rotation matrix representing the robot direction, t is a 3×1 translation vector representing the robot position, and inputs 0 and 1 represent the diagonal structure of the matrix. The Lie algebra of SE (3) is the set of all 4×4 matrices of the form:

$$\begin{bmatrix} \omega & \nu \\ 0 & 0 \end{bmatrix}$$

Where ω is a 3×3 skew-symmetric matrix representing the angular velocity of the robot, ν is a 3×1 vector, the wire represents the output line and input 0 represent the block diagonal of the matrix structure.

CONCLUSION

Lie groups play an enormous role in modern geometry, on several different levels. In this paper we discussed about the basics of Lie algebra, Lie group and matrix lie group. The example of both Lie group and Lie algebra are very familiar to us. In particle physics, matrix Lie groups, particularly special unitary groups and special orthogonal groups, play critical roles in modeling the symmetries of subatomic particles. Lie groups are widely used in many parts of modern mathematics and physics.

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