SPECTRAL GRAPH THEORY

Dissertation submitted in the partial fulfillment of the requirement for the the

MASTER'S DEGREE IN MATHEMATICS

By

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DECLARATION

I Fathima A.S hereby declare that this project entitled **'SPECTRAL GRAPH THEORY'**is a bonafide record of work done by me under the guidence of Toby B Antony ,asociate proffesor on contract Department of Mathematics, Bharata Mata College, Thrikkakara and this work has not previously formed by the basis for the award of any academic qualification, fellowship or other similar title of any other University or Board

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CERTIFICATE

This is to certify that the project entitled **SPECTRAL GRAPH THEORY** submitted for the partial fulfillment requirement of Degree in Mathematics is the original work done by Fathima A.,S during the period of the study in the Department of Mathematics, Bharata Mata College, Thrikkakara under my guidance and has not been included in any other project submitted previously for the award of any degree.

Toby B Antony
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Place:Thrikkakara
Date:

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INTRODUCTION

Spectral graph theory is the branch of mathematics that studies the properties of a graph in relationship to the characteristic polynomial,eigen values,and eigen vector of matrices associated with graph and spectrum of its adjacency matrix or Laplacian matrix.Initially, spectral graph theory examined a graph's adjacency matrix, particularly its eigenvalues. Many advancements over the past 25 years have tended to be more geometric in nature, such as random walks and quickly shifting Markov chains.

Deducing the fundamental characteristic and structure of a graph from its invariants is a key objective of a graph theory. Nearly all important network invariants have strong ties to the eigen values. They include a plethora of graph related knowledge. This is the focus of spectral theory.

There is a lengthy history of spectral graph theory. The 1950s and 1960s saw its emergence. The relationship between graph theoretical study and research in quantum chemistry,which wasn't found until much later,was another important source. Additionally,graph spectra naturally appear in a veriety of theoretical physics topics. Eigen value application can be found in many context and under several names. However it is possible to think of the underlying mathematics of spectral graph theory as a single,cohesive item.

We aim to investigate the core of graph theory in this project. We start out by defining some fundamental graph theory terms. The spectra of a few fundamental graphs are then determined when we introduce the adjacency and Laplacian matrices. We also examine several spectral characteristic and related theorems. The paper ends with a few intriguing spectral graph theory applications.

Chapter 1

PRELIMINARIES

1.1 GRAPH

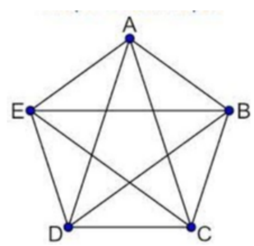
Definition 1.1 *A graph G is a pair (V,E) where V is a non- empty set whose elements are called the vertices of G and E is a sub set of V whose elements are the edges of G and a set of edges,E,where each edge is an unordered pair of vertices.*

Graphs can be represented pictorially as a set of nodes and a set of lines between nodes that represent edges. We say that a pair of vertices,V_i and V_j are adjacent if $V_i, V_j \in$ E where V_i, V_J represents the edge between them. Any two adjacent vertices are neighbours and the set of all neighbours of a vertex v in G is denoted by N_G(V). This graph is an undirected graph. A directed graph is similar to undirected graph except that each edge is assigned an orientation and so we have E $\subset V \times V$.

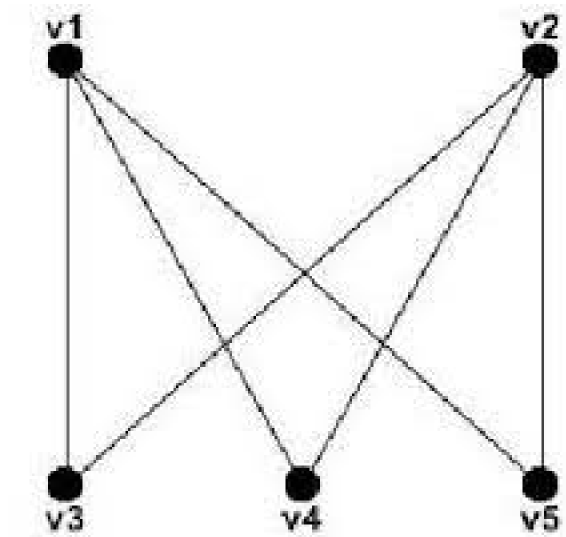
Definition 1.2 *Cardinality of the vertex set is called order of G and the* *cardinality of the edge set is called the size of G. A graph is finite if both its vertex set and edge is finite.[3]*

Definition 1.3 *A loop is an edge that is incident with the same vertex twice. Parallel edges occur when there are two or more distinct edges between the same two vertices. A graph is said to be simple if it has no loop or parallel edges.*

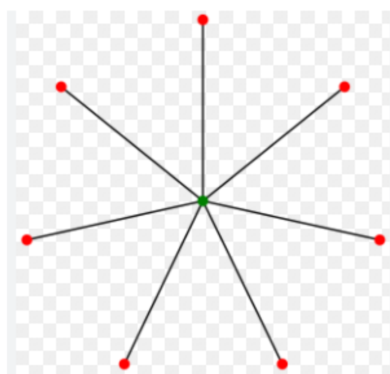
Definition 1.4 *A complete graph, often known as a clique, is a simple graph with any two vertices adjacent. It is denoted by* K_n.



Definition 1.5 *If a graph's vertex set can be divided into two subgroups,* X *and* Y*, with each edge having one end in each subset, that graph is said to be bipartite. A bipartition of the graph and* X *and* Y *is what is referred to as such a partition* (X, Y)*.* G *is referred to as complete bipartite if it is a simple bipartite graph with the bipartition* (X, Y) *such that every vertex in* X *is connected to every vertex in* Y*. It is denoted by* $K_{n,n}$*[3]*



Definition 1.6 *A Star is a complete graph with*$|X|$ *is one or*$|Y|$ *is one.Given below is an example of complete graph,a complete bipartite graph,and star.*



Definition 1.7 *The number of edges in a graph G that are incident with a vertex, denoted as de(v), represents the degree of that vertex. Each loop in the graph is counted twice.*

especially if G is a straightforward graph. The number of neighbors is de(v) in G of v.

A graph G is k regular if d(v) = k; $\forall$ v $\in$G . A regular graph is the one that is*k regular* for some k. For example, the complete graph on n vertices is (n-1) regular.

Definition 1.8 *An alternately placed series of edges and vertices that starts and ends with a vertex is referred to as a walk, n edges make up a walk of length n. A walk is referred to as a trail if all of its edges are distinct. a path is referred to as having all of its vertices being distinct.*

And so a path is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence and are non adjacent otherwise. Likewise, a *cycle* on three or more vertices is a simple graph whose vertices can be arranged in a cyclic sequence such that two vertices are adjacent if they are consecutive in the sequence and are non adjacent otherwise.[1]

The length of a path or a cycle is determined by the number of its edges. We denote by P, the path on n vertices and by C , the cycle on n edges.

Definition 1.9 *Every pair of vertices in graph G is referred to as being connected if a path connects them. The term "connected component" or "component of G" refers to a maximally connected subgraph of G. Thus* *a disconnected graph has atleast two components.*

The *distance* between two vertices u and v of G is defined as the length of the shortest path from u to v.

The *eccentricity* of a vertex is given by e(v)=max$\{$d(u,v):u$\in$ V,u$\neq$ v$\}$ and diameter of a graph G is the maximum of the eccentricities of the vertices in G.

Definition 1.10 ***Empty graph****: An empty graph on n vertices consists of n isolated vertices with no edges. The empty graph on n vertices is the graph complement of the graph* K_n

Line graph: Let G be a loopless graph. Its line graph, L(G) is aa graph whose vertex set is in 1-1 correspondence with the edge set of G and two vertices of L(G), and 2 vertices of L(G) are adjacent iff the corresponding edges in G are adjacent.

A graph is completely defined by either their adjacencies or its incidence, this information can be conveniently stated in matrix form. Indeed, with a given graph,adequately labled,there are associated several matrices.

We shall now define the adjacency matrix and the incidence matrix of a graph. Without loss of generality assume that G is a graph with vertex set V= {1,2,...,n}

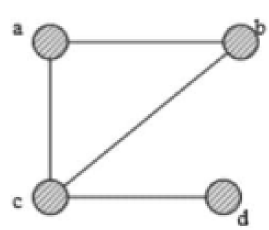
1.2 Adjacency Matrix

Definition 1.11 *The adjacency matrix of a graph G is an n x n matrix* $A_G = (a_{ij})$ *where.for i,j* $\in$ *1,2,..n*

$$a_{ij} = \begin{cases} 1, & i,j \in E \\ 0, & otherwise. \end{cases}$$

The i,j th entry of the matrix denotes the number of edges joining the vertices i and j. Also,we count each loop twice.

For example consider the graph G:



The adjacency matrix is given by $A_G =$

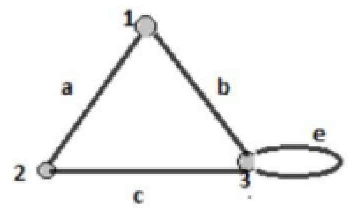
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Clearly, for any graph we may write the adjacency matrix in different ways, by changing the ordering of vertices. We observe that $a_{i,j}$ =a$_{j,i}$ $\forall$ i,j in the adjacency matrix. Thus, the adjacency matrix of any graph is symmetric. In addition if G is simple, then every entry in A_G will be either 0 or 1 with 0s on the diagonal.

1.3 Incidence Matrix

Definition 1.12 *Let G be a graph with vertex set $V = \{1,2,\ldots,n\}$ consisting of m edges. The incidence matrix of G is an $n \times m$ matrix $I_G =(b_{ve})$ where b_{ve} denotes the number of times the vertex v is incident with edge e.*

Consider the graph:



The incidence matrix is given by

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

Theorem 1.1 *In a graph G with vertex set V ={1,2,...,n} A be its adjacency matrix. The entry $A^n_{i,j}$ of the matrix A equals the number of i-j walks oflength n.*

Proof: The proof follows by induction, When n=1, the entry $a_{i,j}$ of A is the number of edges between i and j which is nothing but the number of walks in G of length 1.

Now assume that the result holds for n-1. We have,$A^n = A^{n-1}A$ So,

$$a^n_{i.j} = \sum_{k=1}^{n} a^{n-1}_{i,j} a_{i,j} \mathrm{a}_{i,j} = \begin{cases} 1, & (i,j) \in E \\ 0, & otherwise. \end{cases}$$

it follows that aik aj represents the number of those i-j walks that are formed by joining the edge (k.j) to the i-k walks of length n-1 in G. therefore $A^{n-i}A$ is summation of all walks of length n-1 from vertex I to any other vertex that is adjacent to j. this is equivalent to the number of n walks from i to j. Thus the theorem holds by induction.

Theorem 1.2 *Trace of A^2 is twice the number of edges in the graph.*

Proof: Let G be a graph with order n and size m. Let A be the adjacency matrix and let v_i be an arbitrary vertex of G. For every edge v_iv_j of G, by theorem 1 we get a count 1 towards the (i,i) th position of A^2. That is, the (i,i) th position of $A^2$is equal to the degree of v_i since every v_iv_j walk of length 2 will be a single edge v_iv_j of G. Hence,

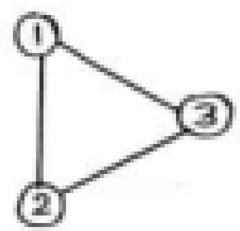
Tr(A^2)= $\sum_{v \in v(G)} d(\mathrm{v}_i) = 2\mathrm{m}$.

Theorem 1.3 *Trace of A^3 is six times the number of triangles in the graph.*

Proof: Let G be a graph with order n and size m, A be the adjacency matrix of G and let v_i be an arbitrary vertex. For every 3 cycle $v_iv_jv_kv_i$ of G, by theorem 1. we get a count of I towards the (i,i) th position of A^3. That is, the (i, i)th position of A^3 which denotes the number of walks of length 3 starting and ending at v_i is equal to the number of 3 cycles that start and end at v_i. Since the vertices of the 3 cycle $v_iv_jv_kv_i$ can be ordered in six ways, we can conclude that trA^3 is six times the number of triangles in G.

1.4 Characteristic Polynomial and Eigenvalues

The characteristic polynomial,Γ of a graph of order n is the polynomial Det (λI-A) where I is the nidentity matrix and A is the adjacency matrix of the graph.
consider the graph,



we have the adjacency matrix as

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

the characteristic polynomial is Γ = det(λI-A) =

$$\begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix}$$

= λ^3-3λ-2

The characteristic polynomial has the general form $\lambda^n + c_1\lambda^{n-1} + ... + c_n$.This polynomial enormously important to spectral grapy theory since it is an algebraic construction that contains graphical informations.The roots of the characteristic polynomial are called *Eigen values.* The Eigen value or characteristic value of matrix A is a scalar λ such that there is a non zero vector x satisfying Ax=λx.

Any non zero vector x satisfying Ax = λ x is called an *Eigen vector or characteristic vector* of A associated with the characteristic value λ.

In the above example if we equate the characteristic polynomial to zero ie.λ^3-3λ-2,and solve we will get the eigen values as $\{1,-1,2\}$.

Chapter 2

THE SPECTRUM OF GRAPHS

Definition 2.1 *The set of a matrix's eigenvalues and their multiplicities is known as the spectrum of the matrix. The spectrum of an adjacency matrix equals the spectrum of the graph. The adjacency matrix's eigenvalues and eigenvectors are therefore regarded as the graph's eigenvalues and eigenvectors.*

Suppose G is a graph.Let $\{\lambda_1,\lambda_2,...,\lambda_t\}$ be the eigenvalues of its adjacency matrix with multiplicities $m_1,m_2,...,m_t$ respectively
Then we write

Spec(G)= $\{ \lambda_1^{(m_1)},\lambda_2^{(m_2)},...,\lambda_t^{(m_t)}\}$ or

Spec(G)=

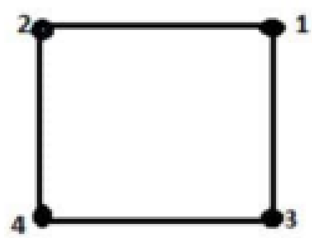
$$\begin{pmatrix} \lambda_1 & \lambda_2 & ...,\lambda_t \\ \mathrm{m}_1 & m_2 & ...\mathrm{m}_t \end{pmatrix}$$

2.0.1 Result

A graph with n vertices have n eigen values.

Proof: This is a direct consequence of the algebraic fundamental theorem. The characteristic polynomial of a graph with n vertices is a polynomial of degree n, and according to a fundamental algebraic theorem, any polynomial of degree n in the area of complex numbers has n roots, counting multiplicity. The graph consequently has n eigenvalues.

For example,consider the graph



Here , A=

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial of the adjacency matrix is λ^4-4λ^2 and the eigen values are {0,0,2,-2}. The graph has three distinct eigen values,-2,2,0.

Spec(G)= {-2,$0^{(2)}$, 2}.

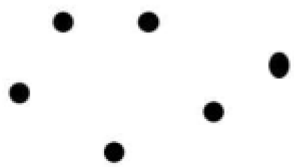
2.1 Spectrum Of Some Fundamental Graphs

In this section we determine the spectra of some fundamental graphs.All the graph considered are find,undirected and simple.

2.1.1 The Graph($\overline{Kn}$)

It denotes the empty graph on n vertices.

$\overline{K_6}$



Since the graph has no edges,adjacency matrix is,

A=

$$\begin{bmatrix} 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{bmatrix}$$

In this case if we consider any non zero vector X in R^n,the equation Ax= λx will imply that λ=0.Thus λ=0 is the only possible eigen value of A.Since the graph has n vertices it must have n eigen values.Hence we can conclude that zero is the only eigen value of ($\overline{Kn}$) with multiplicity n.

Sp($\overline{Kn}$) = $\{0^n\}$

2.1.2 The Complete Graph (Kn)

In a complete graph,each vertex is joined to every other vertex by an edge.So the adjacency matrix will have diagonal entries 0 and all other entries will be 1.

A=

$$\begin{bmatrix} 0 & 1 & 1 & ... & 1 \\ 1 & 0 & 1 & ... & 1 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 1 & 1 & 1 & ... & 0 \end{bmatrix}$$

A is a circulant matrix of order n.

Lemma 2.1 *Let A a circulant matrix of order n with first raw* (a_1,a_2,..,a_n) *then Sp(A)=*{ a_1+$a_2\omega^2$+...+$a_n\omega^{n-1}$,ω= *an nth root of unity.*}

proof: The characteristic poynomial of A is determinant D=det(xI-A).Hence,

D=

$$\begin{bmatrix} x-\mathrm{a}_1 & -\mathrm{a}_2 & ... & -\mathrm{a}_n \\ -\mathrm{a}_n & x-\mathrm{a}_1 & ... & -\mathrm{a}_{n-1} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ -\mathrm{a}_2 & -\mathrm{a}_3 & ... & x-\mathrm{a}_1 \end{bmatrix} \mathit{Let}$$

c_i denote the ith column of D,1≤i≤n and ω,an nth root of unity.Replace C_i by C_1+$C_2\omega$+...+$C_n\omega^{n-1}$.This does not change D.

Let $\lambda_\omega = a_1+a_2\omega+...+a_n\omega^{n-1}$.Then the first new column of D is (x-λ_ω,ω(x-λ_ω),...,$\omega^{n-1}(x-\lambda_\omega))^T$, and hence x-$\lambda_\omega$ isa factor of D.This gives D= $\prod_\omega$(x-λ_ω) and Sp(A) =$\{\lambda_\omega$:ω^n=1$\}$.
Applying this lemma we get,here (a_1,a_2,...,a_n)=(0,1,1,...,1).

$$\lambda_\omega = \begin{cases} n-1, & if,\ \omega = 1 \\ -1, & if,\ \omega \neq 1 \end{cases}$$

hence, Spec(K$_n$)=$\{$(-1)$^{n-1}, (n-1)\}$

2.1.3 Cycle (C_n)

Consider C_n.The adjacency matrix is given by,

A=

$$\begin{bmatrix} 0 & 1 & 0 & ... & ... & ... & 0 & 1 \\ 1 & 0 & 1 & ... & ... & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & ... & ... & 0 & 1 \\ 1 & 0 & 0 & ... & ... & .... & 1 & 0 \end{bmatrix}$$

Clearly A is circulant with first row (010...01). So by lemma 2.2.3, the eigenvalues of A are of the form a_0=$a_1\omega$+$a_2\omega^2$+....+$a_n\omega^{n-1} = \omega+\omega^{n-1}$.
The nth root of unity has the form,ω= $e^{\frac{2\pi ik}{n}}$,k=0,1,...,n-1
$\omega+\omega^{n-1} = e^{\frac{2\pi ik}{n}}+((e^{\frac{2\pi ik}{n}})^{n-1}) = 2\cos\frac{2\pi k}{n}$,k= 0,1,...,n
hence,

$\mathrm{Spec}(C_n) = \{2\cos\frac{2\pi k}{n}; k{=}0, 1, ..., n\}$

2.1.4 Complete Bipartite Graph

Lemma 2.2 *Let G be a graph of order n and size m, and let* Γ*(G, x)* $=x^n+a_1x^{n-1}+...+a_n$ *be the characteristic polynomial of G.then,*

(1)a_1 *= 0*
(2)a_2 *= -m*
(3)a_3 *= -(twice the number of triangles in G)*

Proof: A be the adjacency matrix of G. From direct verification we have$(-1)^r$ = sum of the principal minors of A of order r.
(1) Also we know the principal diagonal entries of A are all 0. Hence result follows directly.
(2) A non vanishing principal minor of order 2 of A is of the form

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

and its value is -1. Also this actually corresponds to an edge of G. hencea_2 =-1×(number of principal minors of the above form) = -m.
(3) A non trivial principal minor of order 3 of A can be one out of the

following three types.

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

and

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

0ut of these only last determinant is non vanishing.Its value is 2 and it corresponds to a triangle in G.This proves (3). Now consider a complete bipartite graph $K_{m,n}$ let V have the partition (X,Y) with $|X|$=m and $|Y|$ = n. then the adjacency matrix of the for
A=

$$\begin{bmatrix} 0 & \mathrm{J}_{m,n} \\ \mathrm{J}_{m,n} & 0 \end{bmatrix}$$

Where $J_{r,s}$ stands for all 1-matrix of size r by s. clearly, rank(A)=2, as the maximum number of independent rows of A is 2. We know,
dim(A)= m+ n = rank(A) + nullity(A).

Hence nullity(A) = multiplicity of eigen value λ=0=m+n-2 i.e. 0 is an eigen value of A repeated m+n-2 times.

Now we can say that the characteristic polynomial of $k_{m,n}$ will be, $\Gamma = \lambda^{m=n-2}(\lambda^2 + Q)$. To find Q we use lemma 2.2.6 and obtain Q = coefficient of $\lambda^{m=n-2}$=-(number of edges) = -mn. $\lambda^{m+n-2}(\lambda^{-2}$-mn)=0 is the characteristic polynomial. λ^{-2}-mn= 0.

$\lambda = \sqrt{mn}$

Hence

Spec$K_{m,n}$ = $\{0^{m+n-2}\sqrt{mn}$,-$\sqrt{mn}\}$

2.1.5 Path(Pn)

We will find a recurrence relationship for the characteristic polynomial of a path graph. Recall that the adjacency matrix of a path graph P. is

$$\text{A(P}_n\text{)=}\begin{bmatrix} 0 & 1 & 0 & 0 & ... & ... & ... & 0 & 0 \\ 1 & 0 & 1 & 0 & ... & ... & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... & ... & 0 & 1 \\ 0 & 0 & 0 & 0 & ... & ... & ... & 1 & 0 \end{bmatrix} Where \text{a}_{i,j}\text{= 1 if } |i-j| = 1\text{,and}$$

$a_{i,j}$ = 0 otherwise.

To find the eigenvalues of the adjacency matrix we need to find characteristic equation of the matrix A(P_n)

$$(\lambda\text{I-A}(\text{P}_n))=\begin{bmatrix} \lambda & -1 & 0 & 0 & ... & 0 & 0 \\ -1 & \lambda & -1 & 0 & ... & 0 & 0 \\ 0 & -1 & \lambda & -1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ \\ ... & ... & ... & ... & ... & ... & ... \\ \\ 0 & 0 & 0 & 0 & ... & -1 & \lambda \end{bmatrix}$$

Thus det(λI-A)= λdet(λI-A(P$_{n-1}$)) $- det$(λI-A(PN-2))

Let f_k be the characteristic polynomial of $A(p_k)$ so that we get the recurrence relation

$f_k = \lambda f_{k-1}$-f_{k-2}.

For P_1,f_1=λ and adjacency spectrum is $\{0\}$

For P_2,f_2 = λ^2-1 and spectrum is $\{$ 1,-1$\}$ For p_3 ,f_3 = λ^3-2λ and the spectrum is $\{0,\sqrt{2},-\sqrt{2}\ \}$

We do not show how to obtain spectrum of path from this equation and instead give result from Godson and Royle.[2]

spec(p_n) = 2cos$\frac{\pi k}{n+1}$,where k=1,2,...,n

2.2 Cospectral Graphs

A question that arises is that can we say that two graphs are isomorphic by simply inspecting their spectra.The answer is no,because there do exist pairs of non isomorphic graphs with the same spectrum.

Definition 2.2 *Graphs with same adjacency spectrum is called cospecral or isospectral graphs.*

Example is given below;

$\mathrm{Spec}(G_1) =$

$$\begin{pmatrix} -2 & 0 & 2 \\ 1 & 3 & 1 \end{pmatrix}$$

$\mathrm{Spec}(G_2) =$

$$\begin{pmatrix} -2 & 0 & 2 \\ 1 & 3 & 1 \end{pmatrix}$$

Thus G_1 and G_2 have same spectrum. But they are not isomorphic.

2.3 SPECTRAL PROPERTIES

The computation of the spectrum was our main focus in the preceding part. We'll now examine how a graph's attributes and spectrum relate to one another. Since the spectrum of a disconnected graph is the union of the spectra of its components, we immediately note that we only need to consider connected graphs.

Definition 2.3 *If the spectrum of a graph G is* { λ_1^{m1},λ_2^{m2},...,λ_k^{mk} } *its minimal polynomial is* $m(G{:}\lambda) = \prod_{i-1}^{k} (\lambda\text{-}\lambda_i)$ *and* $m(G{:}A) = 0$*.And the degree of minimal polynomial is the number of distinct eigen values oh G .*

Lemma 2.3 *A graph G has 1 distinct eigen value if and only if G has no edges.*

Proof: Assume G only has one unique eigenvalue. The adjacency matrix will be a scalar multiple of the identity matrix since the characteristic polynomial of G has degree I in that case. G will therefore be edgeless. In contrast, G will only have one eigenvalue if it lacks any edges.

Lemma 2.4 *The complete graph G is determined by its spectrum.*

Proof: According to our knowledge, a graph with n vertices has n eigenvalues, and the complete graph is the only connected graph with precisely two different eigenvalues. Therefore, if a graph's spectrum contains precisely two different eigenvalues, we can say that the graph is complete. The number of graph vertices is determined by the eigenvalue multiplicities

Chapter 3

THE LAPLACIAN MATRIX OF A GRAPH

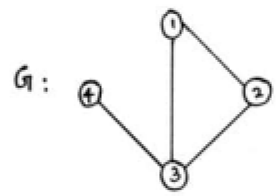
Another significant matrix connected to a graph is the Laplacian, and the Laplacian spectrum is the spectrum of this matrix. The Kirchhoff matrix and the information matrix are two more names for the Laplacian matrix. In a manner similar to the spectral graph theory explored in the preceding parts, we will look at the relationship between a graph's structural characteristics and the Laplacian spectrum in this section.

3.1 Definition

The Laplacin matrix L(G)=($L_{i,j}$) of a graph G on n vertices is an n×n matrix whose entries are given by,

$$L_{i,j} = \begin{cases} d(i) & if i{=}j \\ -1 & if i{\neq}\text{j} \\ 0 & otherwise \end{cases}$$

For example consider G



The Laplacian matrix of G is

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

We can define the degree matrix of G as n×n diagonal matrix. D= $(m_{i,j})$,where,

$$m_{i,j} = \begin{cases} d(i) & if i{=}j \\ 0 & otherwise \end{cases}$$

then we can define laplacian matrix L as L=D-A.

3.2 Properties Of the Laplacian matrix

1.A graph's Laplacian matrix is symmetric and has real entries. L=L^*, and it is hence self adjoint. Furthermore, we are aware that a self adjoint matrix's eigenvalues are actual. Therefore, all of the Laplacian matrix's eigenvalues are real.
2.The row sums as well as the column sums of the Laplacian matrix are 0. This is because when we consider any row or column the diagonal entries of the matrix are degrees of the vertices and every incident edge contributes -1.
3.0 is always an eigenvalue. This follows from the fact that all the row sums are 0 and all the 1s vector is eigen vector for the eigen value 0.
4.The laplacian matrix is positive semidefinite.
A matrix M of order n×n is said to be positive semi definite if ,
X^TX $\geq$ 0.$\forall$x $\in$ R^n.
Here we have X^TX $= \sum_{i,j} X_i \; l_{i,j} \; X_j = \sum_{i,j} l_{i,j} \; X_i \; X_j$
$= \sum_i$ d(i)$l_{i,j}$ X_i^2+2$\sum_{(i,j)\in E} l_{i,j} \; X_i \; X_j + \sum_{i,j\notin E} l_{i,j} \; X_i \; X_j$
$= \sum_i$ d(i)X^2+2$\sum_{i,j\in E}$ (-1)X$_i X_j + \sum_{i,jE} 0_{Xi,Xj}$
$= \sum_{i,j\in E}$(X$_i$-$X_j)^2 \geq$0,$\forall x \in$R^n
Every eigenvalue of a graph G's Laplacian matrix is positive. Suppose L has an eigenvalue of λ, and the related eigenvector x $\in$ R is non zero, L x =λ x.
Afterward, x^TLx=$x^T\lambda$x = λ $||x||^2$ We need to have $\lambda \geq 0$ sincex^TLx$\geq$ 0 and $||x||^2$>0Every eigenvalue of a graph G's Laplacian matrix is pos-

itive.

$\sum_{i}=_{1}^{n}\lambda_i$=trace($L_G$)= sum of degrees of vertices in G.=2m,where m is the number of edges of graph G .

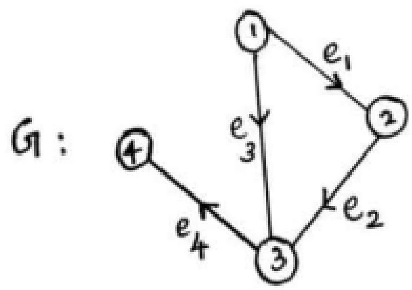
$$\sum_{i=1}^{n}\lambda_i^2 = \text{trace}(L_G^2)=\sum_{i=1}^{n}[d(i)^2+d(i)]=2m+\sum_{i=1}^{n}d(i)^2$$

We can also define Laplacian as follows,
Assume that the graph G (V,E) has an edge set E={e1, e2,..., em| and a vertex set V={V1, V2,..., Va}. Select one of the vertices v, or v, as the positive "end" of e and the other as the negative "end" for each edge ej= (V, Vk). So, each edge in G now has an orientation. The outcome is unaffected by the orientation that is randomly assigned to each edge of G in this case.

The incident matrix ,afforded by the orientation is called the signed incident matrix of G.And it is defined as,

$$q_{i,j} = \begin{cases} 1 & if \text{v}_1 \text{ is the positive end of } e_1 \\ -1 & if \text{v}_1 \text{ is the negative end of } e_1 \\ 0 & otherwise \end{cases}$$

Then the laplacian matrix is given by L=QQ^T.Consider the folloing graph,



We have assigned an orientation to each edge of graph as shown in the figure,the tail of each arrow is chosen as the negative end and head as the positive end.

Q=

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$Q^T =$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

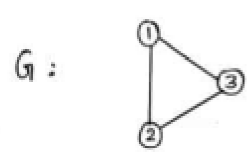
$QQ^T =$

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$QQ^T = L$

3.3 Laplacian spectrum of some graphs

Consider the graph,



G is 2 regular and,

A =

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Its eigen vsalues are λ_1 = -1,λ_2 = -1,λ_3 = 2.
the laplacian matrix is L = 2I-A =

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

det(L-$\tilde{\lambda}$I) = 0
$\tilde{\lambda}^3$-6$\tilde{\lambda}^2$+9$\tilde{\lambda}$ = 0
$\tilde{\lambda}_1$ = 1,$\tilde{\lambda}_2$ = 3,$\tilde{\lambda}_3$ = 0.

Hence we have the result,If G is a k regular graph with adjacency matrix A and laplacian matrix ,then L=D-A =KI-A . Let λ_1,λ_2,...,λ_n be the eigen values os A.Then the eigen values of L will be k-λ_1,k-λ_2,...,k-λ_n.

3.3.1 The complete graph

The complete graph is (n-1) regular.We had seen that ,
Spec(K$_n$)={(-1)$^($n - 1), (n - 1)$}
$SpecL(K_n)$={0,(n)$^{(n-1)}$ }
We see that for the complete graphs, all non zero eigen values coincide.The greatest is n which is also the graph order.

3.3.2 The cycle (C_n)

C_n is a 2 regular graph.We have, Spec(C_n) = $\{2\cos\frac{2\pi k}{n}$;k=0,1,...,n$\}$
SpecLC_n = $\{$ 2-2cos$\frac{2\pi k}{n}$;k=1,2,...,n$\}$

Theorem 3.1 *Let G be a graph with V (G)=$\{$ 1,2,...,n$\}$. Let W be a nonempty proper subset of V (G). Then the determinant of L(W/W) equals the number of spanning forests of G with W components in which each component contains one vertex of W.[4]*

Proof. Assign an orientation to each edge of G and let Q be the incidence matrix. We assume, without loss of generality, that W = $\{$1,2,...,k$\}$. By the Cauchy-Binet formula,

det L(W|W)=$\sum$(detQ[W^c, Z])^2

where the summation is over all Z$\subset$E(G) with $|Z|$=n-k.

Since by Lemma Q is totally unimodular, then ($detQ$[W^c,Z])^2 equals 0 or 1.

Thus, det L(W|W) equals the number of nonsingular submatrices of Q with row setW^c

In view of the discussion in Section Q[W^c, Z] is nonsingular if and only if each component of the corresponding substructure is a rootless tree. Hence, there is a one-to-one correspondence between nonsingular submatrices of Q with row set We and spanning forests of G with |W| components in which each component contains one vertex of W.[4]

Theorem 3.2 *Let G be a graph with V (G)= {1,2,...,n}. Then the co-factor of any element of* $L(G) equals the number of spanning trees of G.$[4]

Proof. Setting W={1} in the above theorem it follows that det L(1|1) equals the number of spanning forests of G with one component, which is the same as the number of spanning trees of G. By Lemma all the cofactors of L(G) are equal and the result is proved.[4]

3.4 ENERGY OF GRAPHS

Definition 3.1 *Energy of graph is defined as the sum of absolute values of the adjacency matrix of the graph.*

Chapter 4

APPLICATIONS

4.1 Epidemic Spread In Networks

Graphs can be used to explore the issue of virus spread and are frequently utilised as abstract representations of various networks. According to the susceptible. invective-susceptible (SIS) model, each node of a network (vertex of a graph) may be in one of two states: healthy yet susceptible to infection (S) or infected (I), with the latter having the potential to propagate illness to susceptible nodes in the network. A person who is sick can also be treated locally, making them susceptible once more. A rate of infection is connected with each edge, and a virus curing rate of six is associated with each infected node. A directed edge from node i to node j indicates that i can infect.The epidemic threshold of a graph G i the value τ such that if $\frac{\beta}{\delta}<\tau$ then the viral outbreaks dies out over time and if not the infection survives.let G be a graph representing this network.During each time interval , an infected node I tries to infect its neighbour's with probability β . At the same time node I can be cured with a probability δ.

Then, the epidemic threshold of G equals $\frac{1}{\lambda_1(G)}$. where λ_1(G) denotes the largest eigen value of the adjaceny matrix of G. An infected node I attempts to infect its neighbours with probability at each time period. Additionally, there is a chance of curing node I.

4.2 Graph Coloring

Vertex-coloring, or the process of giving different colours to a graph's vertices so that no two adjacent vertices have the same colour, is one of the fundamental issues in graph theory. The goal is to utilise the fewest colours feasible. The fewest colours required for such a split is known as the correct chromatic number of a graph G.$\chi_(G)$ is the least number of colours required for such a partition. The graph's spectrum provides us with information on the chromatic number, which is at the core of graph colouring. The boundaries of G's chromatic number are determined by if G has n vertices. ,
$1+\frac{\lambda_1}{-\lambda_n} \leq \chi_(G) \leq 1+\lambda_1$

4.3 Spectral clustering

An essential part of the study of electrical network connections is cluster identification. This is where the graph spectral approach comes in very handy because it allows us to quickly compute the results we require. Edge weights serve as entries in an adjacency matrix. The weights are$\frac{1}{d_{i,j}}$d$_{i,j}$ represented by is the distance between vertices i and j. The objective is to locate the vertices so that the weighted sum of the squared distances between them is as small as possible. The entire procedure won't be covered here, but the main point of importance is that the clustering sites in the graph are determined by the vector component of the Laplacian matrix's second-smallest eigenvalue. The second smallest eigen value for the clustered vertices is the same. Also, only one of the clusters is represented by the biggest eigen value.

4.4 Information Technology

About 10 years ago, it became apparent that graph spectra have numerous significant applications in computer science. Internet technologies, pattern recognition, computer vision, data mining, multiprocessor systems, statistical databases, and many other fields all make use of graph spectra.

The eigenvectors of the adjacency and several related graph matrices provide the foundation of web search engines. Web sites serve as the vertices of a digraph G that depicts the structure of the Internet, while hypertext links between pages serve as its arcs. The Hyperlinked Induced Topics Search (HITS) algorithm takes advantage of the eigenvectors that correspond to the greatest eigenvalues of the matrices AA^T and AA^T, where A is the adjacency matrix of a subgraph of G induced by the collection of websites that certain heuristics have identified from the set of search-keyword-derived webpages. A particular ordering of the chosen webpages is determined by the obtained eigenvectors. In modelling viral spread in computer networks, the adjacency matrix's biggest eigenvalue λ_1 of A is crucial. The resilience increases with decreasing greatest eigenvalue

CONCLUSION

The primary purpose of spectral graph theory is to connect essential structural aspects of a graph to their eigenvalues. It is a developing branch of mathematics having several applications in various fields of study. The major goal of this study was to teach some of the fundamental concepts of spectral graph theory, with a particular emphasis on how to discover the spectra of various types of graphs. We also talked about the Laplacian matrix that is connected with a graph. There is a lot more to say about Laplacian spectrum.

In fact, it is outside the oversight of this work to conduct a "complete" survey of all the literature on the Laplacian spectrum. We also showed how the biological, and computer sciences can benefit greatly from the spectrum of graphs. Spectral graph theory can be used to model a variety of real-world issues. There are numerous sciences and other fields where linear algebra, graph theory, and the spectrum of an are used.

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