CONTROL THEORY

Dissertion submitted in the particle fulfillment of the requirement for the

MASTER'S DEGREE IN MATHEMATICS

By

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DECLARATION

I Serin Isac hereby declare that this project entitled 'CONTROL THEORY'is a bonafide record of work done by me under the guidance of Dr. Lakshmi C,Head Of the Department,Department of Mathematics,Bharata Mata College,Thrikkakara and this work has not previously formed by the basis for the award of any academic qualification,fellowship or other similar title of any other University or Board .

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CERTIFICATE

This is to certify that the project entitled CONTROL THEORY submitted for the partial fulfilment requirement of Master's Degree in Mathematics is the original work don by Serin Isac during the period of the study in the Department of Mathematics,Bharata Mata College,Thrikkakara under my guidance and has not been included in any other project submitted previously for the award of any degree .

Dr. Lakshmi C
Supervisor

Place:Thrikkkakara
Date:

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Contents

ABSTRACT

Control system is simply a mechanism that changes the state of a system.Control theory is a strategy to select proper inputs so as to attain output of our desire.

Chapter 1 consist of basic concepts of control theory.Chapter is divided into 3 sections;First section consist of basic definitions such as open-loop control system and closed-loop control systems;Second section deal with the mathematical model of control systems and third section is about the most modern mathematical model - State Space Model.

The rest of the chapters deal with state space model under consideration and the system under consideration is the LTI systems.Chapter 2 deal with the solution of state equation.Chapter 2 has 2 sections each section dealing with solution of homogeneous and non-homogeneous state equations respectively.

Chapter 3 deals with 2 main concepts of control theory : Controllability and Observability.Chapter 3 has 3 sections with first section dealing with the canonical form of state models and the other 2 sections deal with controllability and observability respectively.The last chapter is about stability of control systems and mainly discusses about stability of LTI systems.

INTRODUCTION

To control an object (or a system) means to influence its behaviour so as to achieve a desired goal.The state of the system determining the way the control has to be exerted at any time can be defined as feedback.Control theory deals with the use of feedback to influence a desired goal.The collection of devices that influences or manage the behaviour of other devices within is known as control system.

Even though control theory has its origin way long before, it was only after 1868 a formal mathematical analysis was given to it. J.C. MAXWELL in 1868 gave a mathematical view of control theory.So the period before 1868 can be called as the prehistory of control, from 1868 to early 1900's it is the primitive period of control, from 1900's to 1960 is the classical period and from then to the present time is the modern era of control theory.

Control theory has major role in many technological advances like spacecraft control, robot technology, smart fluid technology and so on. Control theory can be considered as the linking point of engineering and mathematics. It does plays important role in medical field such as in artificial organs, mechanisms for insulin supply.Some of the engineering fields where control theory has strong connections are Knowledge engineering(deal with knowledge based problem solving along with reasoning) and systems theory and engineering(deal with control,modelling,analysis and design for different real systems).

PRELIMINARIES

Laplace Transfer function
For suitable functions f, the laplace transform is the integral,

$$\mathscr{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt$$

Diagonal matrix

Square matrix with non-zero elements in principal diagonal only.

Eigenvalues

Let B be a square matrix,if $\exists \lambda such that BX = \lambda X$ **for** $X \neq 0$ **the** λ **is the eigenvalue and** $X \neq 0$ **is the eigenvector.If an eigenvalue** λ_i **is repeated ,say,** p **times then B is said to have** λ_i **with multiplicity** p

Non-Singular Matrix-Square matrix whose determinant is non-zero.

Similarity Transformation :

M an N be square matrices.They are similar when,$\exists Q$ a non-singular matrix,such that $Q^{-1}MQ = N$.This transformation is the similarity transformation,where q is the transformation matrix.

Diagonalization of Matrix

Let F be an invertible matrix of order n , D be a diagonal matrix,then $A = FDF^{-1}$ is an $n \times n$ matrix and is said to be diagonalizable.

Quadratic Form-Polynomial of degree two with real or complex coefficients.

Positive-definite Matrix

A symmetric matrix P with real entries is said to be positive-definite if $z^T Pz$ is positive $\forall z \neq 0$(z is the non-zero real column vector).

Chapter 1

Basic Concepts

1.1 Introduction

In simple words a control system can be described as a mechanism that changes the future or state of a system; Control theory is more like a strategy to select proper inputs in order to obtain our desired outputs.A control system usually consists of a plant and an input to the plant. A system of any kind that take an input and make an output is known as a plant and a special kind of input is called a *controller.*

A system of the type that takes an input into the plant and acts on it and produce an output overtime is known as an open-loop control system.Here the input doesn't depend on the output.A washing machine is a good example of an

***open-loop control system*.The goal of a washing machine is to clean clothes;Once we set the time in the machine(which we may note as the input of the system) we get the output regardless of the cleanliness of the clothes.Here the input in the system has no ways to change the output of the system.**

In *closed-loop control system*,also known as feedback control system,the output of the system is measured using a sensor and the result is compared with a reference signal.An error term is generated which is then fed to the controller where it is converted to an input value.

The uncertainty,instability,disturbances(external) of a system are overcome in closed-loop control system.Compared to open-loop control system feedback control system is more efficient.

1.2 Mathematical Model Of Control System

A mathematical model of control system is the representation of the system in terms of mathematical(differential) equations(The input output relations of control systems are

governed by differential equations);by representing the system in mathematical model helps in understanding the system in a better way.Mostly used models are Differential equation model, Transfer function model and State space model.The state space model is the most powerful and modern approach for analysing and designing control systems.

Linearity and Time Invariance

A control system is said to be linear if it obey the superposition and homogenity principle(which states that suppose a model has system response(output) $y_1(t)$ **and** $y_2(t)$ **with respect to the system inputs** $x_1(t)$ **and** $x_2(t)$**,then the system response to linear combination** $a_1x_1(t) + a_2x_2(t)$ **is given by the linear combination** $a_1y_1(t) + a_2y_2(t)$ **where** a_1 **and** a_2 **are constants).**
If the differential equation representing the system has constant coefficients or they are the functions of independent variables then that system is said to be *linear*.If the coefficients are constants then the system is *linear time invariant* and if the constants are functions of time then that model is linear time varying.

Linear Time Invariant(LTI) System: An LTI system possess two properties : i)linearity ii)time-invariance.Using LTI system

benefit in the easiness of the mathematical analysis .Also other physical process which are not LTI can also be approximated by the properties of LTI.Now onwards systems under consideration are LTI systems.

Transfer Function Model

This model is usually used for single input-single output systems.A system's transfer function is defined as ratio of laplace transform of output to laplace transform of input under(zero) initial conditions.Since TF models are restricted to single input-single output systems,no other information about the internal state of systems is provided;even the output of the system might also not be stable in some cases.And hence a much more advanced and modern approach were taken,which is the State Space model

1.3 State Space Model

State and State vector : The group of variables that summarizes the whole of system so to obtain the output is

called *state*. *State vector* is the vector containing state variables as its elements.

The state space model of an LTI system is

$$\dot{X} = AX + BU \tag{1.1}$$
$$Y = CX + DU \tag{1.2}$$

Equation(1.1) is known as the state equation and (1.2) is the output equation.Also X is the state vector.$\dot{X}$ is the differential state vector,U is the input vector and the output vector is Y.A is the system matrix,B and C are the input matrix and output matrix.D is the feed-forward matrix(which can be neglected for now).

Suppose $x_1(t), x_2(t), \cdots x_n(t)$ are the state variables that describe the system,the input variables be $u_1(t), u_2(t), \cdots u_m(t)$ and $y_1(t), y_2(t), \cdots y_p(t)$ are the output variables then in a LTI system the linear combinations of the state variables and the input variables gives first derivatives of the state variables given by,

$$\dot{x_1} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + \cdots + b_{1m}u_m$$
$$\vdots$$
$$\dot{x_n} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + \cdots + b_{nm}u_m$$

where a_{ij}'s andb_{ij}'s are constants.This system of equations is represented in equation (1.1).In a similar way the output variables can be represented as the linear combination of state variables and input variables whose matrix representation is given in equation(1.2)

Chapter 2

Solution Of State Equation

For a given input and initial condition ,if the solution of state equation exists ,then it is easily able to find the output from output equation.

Solution of the state equation is obtained in two cases :

homogeneous and non-homogeneous system. In a homogeneous system no input variable is available and in the latter case solution obtained when input is present.

2.1 Homogeneous System

Consider a scalar differential equation

$$\dot{x} = ax$$

$$\implies \frac{dx}{dt} = ax$$

$$\implies \frac{dx}{x} = adt \implies \int_{x(0)}^{x(t)} \frac{dx}{x} = \int_0^t adt \implies x(t) = x(0)e^{at}$$

Thus solution is $x(t) = e^{at}x(0)$

Now suppose X is a vector and a is the $n \times n$ **constant matrix.Then the state equation is**

$$\begin{aligned} \dot{X} &= AX \\ \implies \frac{dX(t)}{dt} &= AX(t) \\ \implies \int_{X(0)}^{X(t)} \frac{dX(t)}{X(t)} &= \int_0^t Adt \\ \implies X(t) &= e^{At}X(0) \end{aligned}$$

where e^{At} **is known as the state transition or the fundamental matrix.It is denoted by** $\phi(t)$**.Thus for** $\dot{X} = AX, \phi(t) = e^{At}$

There are several methods for obtaining the fundamental matrix such as power series method,Laplace transform method,Diagonalization method and Cayley-Hamilton method.

2.2 Non-Homogeneous System

Scalar Case :

$$\dot{x} = ax + bu$$

$$\dot{x} - ax = bu$$

$$e^{-at}[\dot{x} - ax] = e^{-at}bu$$

$$\frac{d(e^{-at}x(t))}{dt} = e^{-at}[\dot{x} - ax]$$

Upon integration from 0 to t,

$$e^{-at}x(t) - x(0) = \int_0^t e^{-a\theta}bu(\theta)d\theta$$

$$\implies x(t) = e^{at}x(0) = e^{at}\int_0^t e^{-a\theta}bu(\theta)d\theta$$

Vector Case :

$$\dot{X} = AX + BU$$
$$\dot{X} - AX = BU$$
$$e^{-At}[\dot{X} - AX] = \frac{d[e^{-At}X(t)]}{dt} = e^{-At}BU$$
$$e^{-At}X(t) - X(0) = \int_0^t e^{-At}BU(\theta)d\theta$$
$$X(t) = \phi(t)X(0) + \int_0^t \phi(t-\theta)BU(\theta)d\theta$$

Thus, $X(t) = \phi(t)X(0) + \phi(t) * Bu(t)$**.This is arrived due to initial state and input.**

State Transition Matrix

State Transition matrix or Fundamental matrix($\phi(t)$**) is an** $n \times n$ **matrix.**

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$\frac{d}{dt}e^{At} = A + A^2t + \frac{A^2t^2}{2!} + \cdots$$

$$= A(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$= Ae^{At}$$

$$= e^{At}A$$

Properties Of State Transition Matrix : For a system $\dot{X} = AX$,**whose** $\phi(t) = e^{At}$**,**

$$1.\phi(0) = e^{A0} = I$$

$$2.\phi^{-1}(t) = \phi(-t)$$

$$3.\phi(t_1 + t_2) = \phi(t_1)\phi(t_2)$$

$$4.[\phi(t)]^n = \phi(nt)$$

Chapter 3

Controllability,Observability

3.1 Canonical Form Of State Model

Let the system's state equation given by,

$$\dot{X} = AX$$

where X is the state vector.A new state vector,W, can be defined,so that $X = NW$, where N is the diagonalization matrix(or the modal matrix).
Our state model is given by

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

substitute X=NW here which results in,

$$\dot{X} = ANW + BU \tag{3.1}$$
$$Y = CNW + DU \tag{3.2}$$

Equation 3.1 is premultiplied by N^{-1}

$$\implies N^{-1}\dot{X} = N^{-1}ANW + N^{-1}BU$$

differentiate $X = NW$

$$\implies \dot{X} = N\dot{W}$$

pre multiplying by N^{-1},$N^{-1}\dot{X} = \dot{W}$
thus, $\dot{W} = N^{-1}ANW + N^{-1}BU$ let $N^{-1}AN = \Delta$(Δ is known as the grammian matrix).

$$N^{-1}B = \hat{B}$$

$$CN = \hat{C}$$

thus our state model is transformed into,

$$\dot{W} = \Delta W + \hat{B}U$$

$$Y = \hat{C}W + DU$$

The transformed model is known as the canonical form of state model.Note that the diagonalization matrix ,N is obtained by arranging the n eigenvectors.
Note : In the canonical form of state model,A(system matrix) is a diagonal matrix.

Suppose A is a non-diagonal system matrix having different eigen values,then A is converted to a diagonal matrix by similarity transformation using the diagonalizatin matrix,N.

3.1.1 Jordan Canonical Form

When the system matrix A is a non-diagonal matrix,with its eigen values having multiplicity,then the transformation is as

follows:

$$\dot{W} = HW + \hat{B}U$$

$$Y = \hat{C}W + DU$$

where $H = N^{-1}AN$**,**$\hat{B} = N^{-1}B, \hat{C} = CN.$

H **is called the Jordan matrix and it has a Jordan block of size** $f \times f$**,corresponding to an eigen value** λ_i **having multiplicity** f**.Note that in** H **the diagonal elements are the eigenvalues and elements above them are 1.**

2 basic questions in control problems that are required to be answered in order to decide if a control solution exist or not are:
1) Is it possible to transfer the system, in finite time,from the initial state to any desired state with the help of proper control force?
2) Knowing the output vector for a finite time, is it possible to determine the system's initial state?

There are 2 concepts that are required to solve the above questions : Controllability and Observability

3.2 Controllability

A system is called completely controllable when the transformation of system's initial state to any desired state by a control input vector under finite time is possible.

Controllability is tested by i)Kalman's Test ii)Gilbert's Test

Gilbert's Test

There are two cases : i)When A has different eigenvalues ii) When A has eigenvalues of multiplicity.
Case i) : Consider the state model,

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

the canonical form of above is given by

$$\dot{W} = \Delta W + \hat{B}U$$

$$Y = \hat{C}W + DU$$

here system is completely controllable iff there are no rows of $\hat{B}$ that contains all zeros.
Case ii) : In this case the model's transformation is Jordan Canonical.The state model given by

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

transformed model is

$$\dot{W} = HW + \hat{B}U$$

$$Y = \hat{C}W + DU$$

here the system is completely controllable if the elements in any row of $\hat{B}$ corresponding to each Jordan block's last row are not all zeros.

Kalman's Test

Consider the state equation ,$\dot{X} = AX + BU$

Form a matrix Q_m,

$$Q_m = \begin{bmatrix} B & AB & A^2B & \cdots A^{n-1}B \end{bmatrix}$$

where n is the state variables number.

A system is completely controllable if rank of Q_m is n.

3.3 Observability

A system is completely observable when every $X(t)$ is identified(completely) by measuring $y(t)$ under finite time.

2 tests for Observability:

i) Gilbert's Test ii) Kalman's Test

Gilbert's Test

The state model given by

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

the transformed model is either

$$\dot{W} = \Delta W + \hat{B}U$$

$$Y = \hat{C}W + DU$$

or

$$\dot{W} = HW + \hat{B}U$$

$$Y = \hat{C}W + DU$$

The system is completely observable iff no columns of $\hat{C}$ is zero($\hat{C} = CN$).

Kalman's Test

State model is

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

define Q_o by,

$$Q_o = \left[C^T \;\; A^T C^T \;\; \left(A^T\right)^2 CT \;\; \cdots \;\; \left(A^T\right)^{n-1} C^T\right]$$

If rank of Q_o is n,then it is completely observable.

3.3.1 Example

A system given by

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} X + \begin{bmatrix} 1 \\ -1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 1 & 1 \end{bmatrix} X$$

To check controllability : given

$$B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$Q_m = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

rank of $Q_m = 1 \neq 2$ hence by Kalman's test system is not controllable.
To check Observability:

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

$$A^T CT = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$Q_0 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

rank is 1.and So system is not observable.

Chapter 4

Stability

Stability as the name suggest indicates the stable working condition of a system.When the system is stable,the output is finite,predictable and also stable for an input given.

- An autonomous system(LTI system),$\dot{X} = F(X)$ is said to be *stable at the origin*,if for every initial state $x(t_0$,sufficiently close to origin,if $x(t)$ remains near the origin for all t.
- Asymptotically stable : An autonomous system,$\dot{X} = F(X)$ is asymptotically stable if $X(t)$ approaches origin as $t \to \infty$.
- Asymptotically stable in the Large : If its asymptotically stable for every initial condition.

Stability Of LTI System

Equilibrium Point :
Consider the system

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

if for a given input vector v(t),$\exists$ a state variable x such that$\dot{x(t)} = 0 \forall t$,then x is called equilibrium point of the system.

Lyapunov Stability Condition

Let $x = 0$ be equilibrium point for some $\dot{x} = f(x)$.let V a continuously differentiable function so that $V(0) = 0, V(x) > 0 and \dot{V(x)} \leq 0$,then $x = 0$ is stable.Moreover,if $\dot{V(x)} \leq 0$,then $x = 0$ is asymptotically stable.

<u>Stability Of LTI System</u>

Consider the LTI system

$$\dot{x} = Ax$$

Choose $V(x) = x^T Px$.Here P is chosen such that it is real,symmetric and positive definite.Now the time derivative of

$$\dot{V(x)} = (\dot{x})^T Px + x^T P\dot{x}$$

$$(\dot{x})^T = (Ax)^T = x^T A^T$$

thus $\dot{V(x)} = x^T A^T Px + x^T PAx$

$$\dot{V(x)} = x^T (A^T P + PA)x$$

let $Q = -(A^t P + PA)$ then $\dot{V(x)} = -x^T Qx$.

Now since $V(x)$ is chosen to be positive definite, $\dot{V(x)}$ must be negative definite. Thus if Q is positive definite,then given system is stable.So to check stability first specify Q(let it be an identity matrix) and then check whether matrix P obtained by solving $Q = -(A^T P + PA)$ is positive definite.

CONCLUSION

A system is a physical device that generate an output for a given input.Control System is simply a mechanism that can change the state of the system.Control theory is a study about ways to select inputs so as to obtain a desired output

The period from 1920-1950 is known as classical control period.Modern control is after 1950 to till now.There were many drawbacks of classical control such as the design methods in classical control were simply trial and error methods.The model used those days were the transfer function model which dealt only with single input single output systems.So as to achieve these drawbacks a new approach was taken which gave the State Space Model.There are many applications of control theory.Contol theory is considered as the meeting point of engineering and mathematics.

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