MODULAR LATTICE

Dissertation submitted in partial fulfilment of the requirement for the

MASTER'S DEGREE IN MATHEMATICS

By

ARCHANA G S

Reg. No: 210011014836

Under the Guidance of

Dr. LAKSHMI C

Department of Mathematics

BHARATA MATHA COLLEGE, THRIKKAKARA

Affiliated to Mahatma Gandhi University, Kottayam

2021 - 2023

CERTIFICATE

This is to certify that the project entitled,**"MODULAR LATTICE"** submitted for the partial fulfilment requirement of Master's Degree in Mathematics is the original work done by **ARCHANA G S** during the period of the study in the Department of Mathematics, Bharata Matha College, Thrikkakara under my guidance and has not been included in any other project submitted previously for the award of any degree.

Dr.LAKSHMI C

Supervisor

Place : Thrikkakara

Date :

DECLARATION

I, Archana G S, hereby declare that this project entitled " MODULAR LATTICE" is a bonafide record of work done by me under the guidance of Dr. Lakshmi C, Assistant Professor, Department of Mathematics, Bharata Matha College, Thrikkakara and this work has not previously formed by the basis for the award of any academic qualification, fellowship or other similar title of any other University or Board.

ARCHANA G S

M.Sc. Mathematics

Bharata Matha College, Thrikkakara

Place : Thrikkakara

Date :

ACKNOWLEDGEMENT

First and foremost I express my sincere gratitude towards the abiding presence and grace of God Almighty.

I sincerely express my gratitude to Dr. Lakshmi C , Assistant professor, Department of Mathematics, Bharata Matha College, Thrikkakara for guiding and helping me to complete this work.

I also express my profound gratitude to Dr. Lakshmi C , Head of the Department of Mathematics, Bharata Matha College, Thrikkakara for her wholehearted cooperation and valuable advices.

Finally I am obliged to all our teachers of the Mathematics Department, Bharata Matha College for the encouragement for completing this work.

ARCHANA G S

Contents

INTRODUCTION 1

1 Lattice-Basic concepts 2

2 Modular lattices 14

3 Representation of modular lattice 24

4 Cubicles 31

CONCLUSION 40

BIBLIOGRAPHY 41

INTRODUCTION

The notion of lattice was introduced by Richard Dedekind to study the relations between ideals and rings of numbers. The lattice concept helps to unify a number of ideas.This paper aims to give a very basic concept about lattices,modular lattices and representation of modular lattice D.The key idea is the construction of sub lattices B(l) and B(l) called cubicles.We start with the fundamental concepts,properties and propositions about lattice and different lattices including distribute lattice with examples in the first chapter.We then discuss about modular lattice and its characterization in second chapter.Representation of modular lattices are discussed in the third chapter.Eventually we end up with discussing about the sub lattices of modular lattices called cubicles and their construction are discussed in the fourth chapter.

Chapter 1

Lattice-Basic concepts

.

section 1.1.

Definition:Partial order

A partial order is a binary relation < over a set P which is reflexive,antisymmetric and transitive,that is which satisfies for all a,b and c in P;[1]

- a$\leq$ a(reflexivity).
- if $a \leq b$, and ,b$\leq$ a then b=a (antisymmetric)
- a$\leq$ b and b$\leq$ c then a$\leq$ c (transitivity)[1]

A set with a partial order is called a partially ordered set(also called a poset). A poset (L,<) is called a lattice ordered if for every pair x,y of elements of L their supremum and infimum exist.[1]

Examples

- The set R of real numbers ordered by the standard relation $<$ is a poset.
- The set N of natural numbers equipped with the relation of divisibility.That is here a¡b means a/b(a divides b) forms a POSET.
- Let H be a collection of sets A,B,C...then H under contained in forms a poset.i.e, here A¡B means $A \subseteq B$.[1]

Hasse Diagrams

A Hasse diagram is a graphical representation of the relation of elements of a partially ordered set. To define Hasse diagrams, we first define a relation covers as follows.For any two elements $x,y \in X$,Y covers x if $x<y$ and $\forall\ z \in X$:$x \leq z < y$ implies $z=x$.In other words, there should not be any element z with $x<z<y$. We use $y \mapsto x$ to denote that y covers x(or x is covered by y). We also say that y is an upper cover of x and x is a lower cover of y.A Hasse diagram of a poset is a graph with the property that there is an edge from x to y iff $y \mapsto x$.[1] Furthermore,when drawing the graph on a Euclidean plane,x is drawn lower than y when y covers x.This allows us to suppress the directional arrows in the edges.[1]

Join and Meet Operations

Now define on subset of X-meet or infimum and join or supremum.

Let Y⊆X, where (X;≤) is a poset. For any m∈X, we say that m=inf Y iff

1. ∀y∈Y:m≤y,and
2. $\forall m'$ ∈X:(∀ y ∈ Y ∈Y:m'≤ y) $\implies$ m'≤ m.[1]

.

The condition (1) implies m is a lower bound of Y and (2) implies if m' is another lower bound of Y,then m' is less than m.

Then m is called greatestlower bound of the set Y.Observe that m is not required to be an element of Y.The definition of sup is similar.For any s∈X,we say that s=sup Y iff[1]

1. ∀y∈Y:y≤s
2. ∀s'∈X:(∀y∈Y:y≤s') $\implies$ s≤s'[1]

.

Then s is called the least upper bound of Y.We denote the glb a of $a, b$by a∪b,and lub of a, b by a∩b.In the set of natural numbers ordered by the divides relation ,the glb corresponds to finding the greatest common divisor(gcd)and the lub corresponds to finding the least common multiple of two natural numbers.The greatest lower bound or the least upperbound may not always exist.[3]

Lemma1.1.1[Connecting Lemma]

1. $x \leq y \equiv (x \cup y) = y$,and

2. $x \leq y \equiv (x \cap y) = x$[3]

.

Proof:

$x \leq y$ implies that y is an upper bound on x, y.Y is also the least upper bound because any upper bound of x, y is greater than both x and y.Therefore,$(x \cup y) = y$.Conversely,$(x \cup y) = y$ means y is an upper bound on x, y.Therefore,$x \leq y$.

The proof for the second part is the dual of this proof.

section 1.2.

Definition:Lattices.

A poset (x;≤) is a lattice iff ∀x,y∈X: x∪y and x∩y exist. Or a lattice L is a set with two binary operations, intersection and sum.

if a,b∈L,we denote their intersection by a.b and their sum by a+b.Each of those operations is commutative and associative. Moreover for any a,b∈L we have the identities;[2]

1. a,b∈L,a.a=a and a+a=a (idempotent law)
2. a,b∈L,a.b=b.a and a+b=b+a (commutative law)
3. a,b,c∈L,a.(b.c)=(a.b).c and a+(b+c)=(a+b)+c (associative law)
4. a,b∈L,a.(a+b)=a,a+(a.b)=a (absorption law)[2]

.

In Mathematics a lattice is a partially ordered set in which every two elements have a unique supremum and a unique infimum. An example is given by when P is a set of positive integers set x≤y in P when x divides y without remainder. Lattices can also be characterized as algebraic structure satisfying certain axiomatic identities.Lattice theory draws on both order theory and universal algebra since the two definitions are equalent.[2]

Lattices as partially ordered sets.

If (L,≤)is a partially ordered set (poset) and S⊆L is an arbitrary subset, then an element u∈L is said to be an upper bound of S if s≤u for each s∈S. A set may have many upper bounds or none at all. An upper bound u of S is said to be its least upper bound or join or supremum, if u≤x for each upper bound x of S.A set need not have a least upper bound, but it cannot have more than one.Dually l∈L is said to be lowerbound of S,if l∈s for each s∈S. A lowerbound l of S is said to be its greatest lowerbound or meet or infimum if x≤l for each lowerbound x of S. A set may have many lowerbounds or none at all,but can have at most one greatest lowerbound. [2]

Lattices as algebraic structures

A Join-semilattice is a partially ordered set that has a join for any nonempty finite subset and a meet-semilattice is a partially ordered set which has a meet for any nonempty finite subset , denoted by a∧b and a∨b respectively.(L,≤) is called a lattice.If it is both a join and a meet semi lattice.This definition makes ∧ and ∨ binary operations.[1] An algebraic structure (L,∧,∨) consist of a set L and two binary operations ∧ and ∨ on L is a lattice if the following axiomatic identities hold for all elements a,b,c of L.[3]

- Commutative laws: a∨b=b∨a, a∧b=b∧a
- Associative laws: a∨(b∨c)=(a∨b)∨c a∧(b∧c)=(a∧b)∧c
- Absorption laws: a∨(a∧b)=a, a∧(a∨b)=a[2]

The following two identities are also usually regarded as axioms, even though they follow from the two absorption laws taken together.[1]

- Idempotent laws: a∨a=a, a∧=a

These axioms implies the (L,∨) and (L,∧) are semi lattices. The both join and meet appear in the absorption laws, assure the two semi lattices interact appropriately and distinguish a lattice from an arbitrary pair of semi lattices. In particular each semi lattice is the dual of the other.[2] **Examples**

- Let X be a non empty subset, then the poset(P(X),≤) of all subsets of X is a lattice with the binary operations ∩ and ∪ respectively.[2]
- Let V be a vector space, and L be the set of linear subspace ordered by inclusion. L is lattice with two binary operations intersection as S.T and sum as S+T.[2]
- The set N of all natural numbers under divisibility forms a lattice. Here two operations is defined as a.b=g.c.d(a,b) and a+b=l.c.m(a,b) for all a,b∈N.[2]

section 1.3.

Definition:Sub Lattices.

S is a sub lattice of a given lattice L=(X;≤) iff it is non-empty, and:∀a,b∈S:sup(a,b)∈S∧ inf(a,b)∈S.[3] Note that the sup and inf of any two elements in the sub lattices S must be the same as the sup and inf of those elements in the original lattice L. It is not sufficient that S being a subset of a lattice when S to be a sublattice. In addition to S being a subset of a lattice, sup and inf operations must be inherited from the lattice. Also can be defined as; A sub lattice of a lattice L is a non empty subset of L that is a lattice with the same meet and join operations as L. That is if L is a lattice and M≠∅ is a subset of L such that for every pair of elements a,b in M both a∨b and a∧b are in M, then M is a sublattice of L.[3]

Examples

1. If L is any lattice and a∈L be any element then a is a sublattice of L.
2. For any two elements x, y in a lattice L with x≤y, the interval [x, y]= a∈L/x≤a≤y is a sublattice of L.[3]

Definition:Bounded lattice

A bounded lattice is an algebraic structure of the form (L,∨,∧,1,0) such that L,∨,∧ is a lattice, 0 (the lattices bottom) is the identity element for the join operator , and 1 (the lattices top) is the identity element for the meet operator ∧.[3]

- Identity laws: a∨0=a,a∧1=a

Definition:Complete lattice

A poset is called complete lattice if all its subsets have both a join and a meet.[3]

Definition:Continuous lattice

A continuous lattice is a complete lattice that is continuous as a poset.[3]

Definition:Algebraic lattice

An algebraic lattice is a complete lattice that is algebraic as a poset.[3]

Definition:Graded

A lattice (L,≤) is called graded, sometimes ranked,if it can be equipped with a rank function r from L to N sometimes to Z, compactible with the ordering (r(x)≤r(y) whenever x≤ y) such that whenever y covers x , then r(x)+1=r(y). The value of the rank function for a lattice element is called its rank.[3]

Definition:Distributive lattice

A lattice L is called distributive if the laws x+(y.z)=(x+y).(x+z) and x.(y+z)=(x.y)+(y.z) hold for all x,y,z∈L. These equalities are called distributive laws.[3]

Examples

1. (P(M),∩,∪) is a distributive lattice as A∩(B∪C)=(A∩B)∪(A∩C)

2. Lattice N of natural numbers under divisibility is distributive. Hence a.b=g.c.d(a, b) and a+b=l.c.m(a, b). Thus (N,g.c.d,l.c.m) is a distributive lattice.[3]

Definition:Complementary

Two elements x, y in an interval [a, b] are complementary and call each a relative complement of the other if x.y=a, x+y=b.[3]

Proposition 1.3.1 A lattice L is distributive if the equations a.c=b.c and a+b=b+c imply a=b hold for any a,b,c$\in$L.[3]

proof: Let L be distributive and let a.c=b.c and a+b=b+c. Then a=a.(a+c)=a.(b+c) =(a.b)+(a.c =(a.b)+(b.c) =b.(a+c) =b.(b+c)=b Hence the proposition.[3]

Proposition 1.3.2 In any distributive lattice, relative complements in each interval are unique.[3]

proof: Let [a,b] be an interval. Let y and z be relative complements of x in [a,b]. Then, x.y=a=x.z and x+y=b=x+z. Hence y=z by above proposition.[3]

Theorem 1.3.3

1. Let (L,$\leq$) be a lattice ordered set. If we define x.y=inf(x,y) and x+y=sup(x,y),then (L,.,+) is an algebraic lattice.

2. Let (L,.,+) be an algebraic lattice. If we define x$\leq$y iff x.y=x, then (L,$\leq$) is a lattice ordered set.[3]

proof:

1. Let (L,$\leq$) be a lattice ordered set for all x,y,z$\in$ Lwe have, x.y=inf(x,y)=inf(y,x)=y.x x+y=sup(x,y)=sup(y,x)=y+x Also, x.(y.z)=x.inf(y,z)=inf(x,inf(y,z)) =inf(x,y,z) =inf(inf(x,y),z) =inf(x,y).z =(x.y).z And similarly ,(x+y)+z=x+(y+z) x.(x+y)=x.sup(x,y)=inf(x,sup(x,y))=x x+(x.y)=x+inf(x,y)=sup(x,inf(x,y))=x

2. Let (L,.,+) be an algebraic lattice. Clearly for all x,y,z∈L x.x=x and x+x=x So x≤x. i.e, ≤ is reflexive. If x≤y and y≤x ,then x.y=x and y.x=y and x.y=y.x So x=y. i.e, ≤is antisymmetric. If x≤y and y≤z, then x.y=x and y.z=y Therefore x=x.y=x.(y.z)=(x.y).z=x.z. So x≤z. i.e, ≤is transitive. Let x,y L. Then x.(x+y)=x implies x≤x+y and similarly y≤x+y If z L with x≤z and y≤z,then (x+y)+z=x+(y+z)=x+z=z. And so x+y≤z. Thus sup(x,y)=x+y similarly inf(x,y)=x.y. Hence (L,≤) is a lattice ordered set.[3]

section 1.4.

Some important properties of lattices.

- Completeness

  A poset is called a complete lattice if every subset of poset has a least upper bound and a greatest lower bound. In particular, every complete lattice is a bounded lattice. [3]

  Every poset that is a complete semilattice is also a complete lattice. Related to this result in the interesting phenomenon that there are various competing notions of homomorphism for this class of posets, depending on whether they are seen as complete lattices, complete join semilattices, complete meet semilattices or as join complete or meet complete lattices.[3]

- Conditional completeness.

  A conditionally complete lattice is a lattice in which every non empty subset that has an upperbound has a join.generalization of the completeness axiom of the real numbers given by such lattices. A conditionally complete lattice is either a complete lattice or a complete lattice without its maximum element 1, its minimum element 0, or both.[3]

- Distributivity[3]

  Since lattices come with two binary operations, it is natural to ask whether one of them distributes over the other. i.e, whether one or the other of the following dual laws for every three elements a,b,c of L; Distributivity of over : a (bc)=(ab) (ac) Distributivity of over :a(bc)=(ab) (ac) A lattice that satisfies the first or equivalently, the second axiom, is called a distributive lattice.The only non-distributive lattices with fewer than 6 elements are called M3(diamond lattice) and N5(pentagon lattice).[3] They are shown in figure: 1 and figure: 2 respectively.

- Modularity[3]

  For some applications the disrtibutivity condition is too strong and the following weaker property is often useful.A lattice $(L,\wedge,\vee)$ is modular if, for all elements a,b,c of the following identity holds;

  Modular identity: $(a\wedge c)\vee(b\wedge c)=[(a\wedge c)\vee b]\wedge c$ This condition is equivalent to the following axiom, Modular law: $a\leq c$ implies $a\vee(b\wedge c)=(a\vee b)\wedge c$.[3]

.

Chapter 2

Modular lattices

.

section 2.1.

Modular lattice

We describe a special class of lattices called modular lattices. Modular lattices are numerous inmathematics. In the branch of mathematics called order theory a modular lattice is a lattice that satisfies the following self dual condition; Modular law: x≤b $\Longrightarrow$ x∨a∧b=(x∨a)∧b; where $\leq$ is the partial order, and and are the operations of the lattice.[4] Modular lattice arise naturally in algebra and in many other areas of mathematics. For example, the subspaces of a vector space (and more generally the submodules of a module over a ring) form a modular lattice.Every distribution lattice is modular. Modular elements are the elements in which a is not necessarily modular lattice there is an element b for which the modular law holds in connection with arbitrarily elements a and b . A pair (a,b) may hold the modular law is called the modular pair, and there are various generalisations of modularity related to this notion and to semimodularity. The modular law can be seen as a restricted associative law that connects the two lattice operations similarly to the way in which the associative law, μx=$\alpha\mu$x for vector spaces connects multiplication in the field and scalar multiplication. The restriction x≤b is clearly necessary since it follows for x∨a∧b=x∨a∧b. In other words, no lattice with more than one element satisfies unrestricted consequent of the modular law. (To see this, just pick non maximal b and let x be any element strictly less than b.) [4]

It is easy to see that x≤b implies x∨a∧bx∨a∧b in every lattice. Therefore the modular can be also being stated as modular law (variant):- x≤b implies x∨a∧bx∨a∧b. By substituting x with xb, the modular law can be expressed as an equation that is required to hold unconditionally, as follows: Modular identity:-x∧ba∧b=a∧b∨a∧b. The smallest non modular lattice is the "pentagon" lattice N5consisting of five elements 0, 1, x, a, b such that, 0<x<b<1, 0<a<1, and a is not comparable to x or to b. For this lattice x∨a∧b= x∨0= x<b=1∧b =x∨a∧b holds, contradicting the modular law. Every nonmodular lattice contains a copy of N5as a sublattice. Modular lattice are sometimes called Dedekind lattices after Richard Dedekind who discovered the modular identity in several motivating examples.[4]

Definition:Modular

A lattice L is called modular (or Dedekind) if for every x,y,z∈L such that x≤z, we have x+(y.z)=(x+y).z This relation is called Dedekind's axiom.[5]

Lemma 2.1.1

For all lattices, a∩(b∪c)≥(a∩b)∪(a∩c)[5]

proof:

a∩(b∪c)≥(a∩b)

a∩(b∪c)≥(a∩c) Combining, we get

a∩(b∪c)≥(a∩b)∪(a∩c) A similar observation follows from the definition of modular lattices.[5]

Lemma 2.1.2

For all lattices,a≥c $\Longrightarrow$ acap(b∪c)≥(a∩b)∪c[5]

proof:

a≥c $\Longrightarrow$ a∩c=c Using Lemma 2.1.1it follows that a∩(b∪c)≥(a∩b)∪c[5]

Lemma 2.1.2

L is distributive implies that L is modular.[5]

proof:

Assume that L is distributive. Then by f definition of a distributive lattice , ∀a,b,c:a∩(b∪c)=(a∩b)∪(a∩c) $\Longrightarrow$ ∀a,b,c:a≥c implies a∩(b∪c)=(a∩b)∪c (since a≥c is equivalent to a∩c=c.)

Therefore L is modular.[5]

Duality principle[2]

Any formula involving the operations .and + which is valid in any lattice (L,.,+) remains valid if we replace . by + and + by . everywhere in the formula. This process of replacing is called dualizing.[2]

Examples[2]

1. The normal subgroups of a group ordered by set inclusion from a modular lattice.

   Let G be any group and L be the set of all normal subgroups of G. Then L$\neq\phi$ as G$\in$L. (L,$\subseteq$) is then a poset.

   For any A,B$\in$L, let A.B=A$\cap$B and A+B=AB Also A$\subseteq$AB and B$\subseteq$AB as a$\in$A $\Longrightarrow$ ae$\in$AB,etc.

   Thus AB is the smallest normal subgroup containing A and B. Indeed if C is any normal subgroup containing A and B, then AB$\subseteq$C (x$\in$AB $\Longrightarrow$ x=ab$\in$C as a$\in$A$\subseteq$C, a$\in$A$\subseteq$C.

   Now to check the modularity conditions. Here we proceed by applying the Duality principle.

   Let A,B,C$\in$L with A$\supseteq$B be any members.

   We show A.B+C=B+(A.C)

   i.e,A$\cap$BC=B(A$\cap$C)

   Let x$\in$A$\cap$BC be any element.

   Then x$\in$A and x$\in$bc $\Longrightarrow$ $\exists$ b$\in$B, c$\in$C s.t,x=bc x$\in$A $\Longrightarrow$ bc$\in$A. Also b$\in$B$\subseteq$A $\Longrightarrow$ b-1$\in$A Thus b-1bc$\in$A $\Longrightarrow$ c$\in$A $\Longrightarrow$ c$\in$A$\cap$C.

   So b$\in$B, c$\in$A$\cap$C $\Longrightarrow$ bc$\in$ BA$\cap$C $\Longrightarrow$ x$\in$BA$\cap$C $\Longrightarrow$ A$\cap$BC$\subseteq$B(A$\cap$C)

   Again if y$\in$B(A$\cap$C) Then y=bk ;where b$\in$B, k$\in$A$\cap$C

   Now b$\in$B$\subseteq$A, k$\in$A $\Longrightarrow$ bk$\in$A

Also b∈B, k∈C $\implies$ bk∈BC Thus bk∈A∩BC $\implies$ B(A∩C)⊆A∩BC

Hence, B(A∩C)=A∩BC.

2. Let $< R, =, . >$be a ring and L be the set of all ideals of R, then (L,⊆)forms a modular lattice.

   (L,⊆) forms a lattice where for any A,B∈L,A.B=A∩B and A+B=A∪B . We show that L is modular.

   Let A,B,C∈L be any three members with A⊆B.

   We claim A∩(B+C)=B+(A∩C)

   Let x∈A∩(B+C)be any element. Then x∈A and x∈B+C $\implies$ x∈A and x=b+c,b∈B,c∈C Now b∈B⊆A, x=b+c∈A Thus b+c-b∈A $\implies$ c+b-b∈A $\implies$ c∈A $\implies$ c∈A∩C

   i.e, x=b+c,b∈B,c∈A∩C $\implies$ x∈B+A∩C

   i.e, A∩(B+C)⊆B+(A∩C)

   Again by modular inequality which holds in every lattice,

   A∩(B+C)⊇B+(A∩C)

   Hence,A∩(B+C)=B+(A∩C)

   $\implies$ L is a modular lattice.[2]

proposition 2.1.4

A lattice L is modular if and only if, for each interval I of L,any two elements of I which are comparable and have a common complement in L are equal.[3]

proof:

In a lattice L, given a,b,c∈L, if a≤c then a+(b.c)=a+band a+(b.c)≤c

From the identities of lattice we have a.b=b and a+b=b fora,b∈L.

Hence b.c=b. Therefore a+(b.c)=a+b

Now, a+(b.c)=a+b=b+a=a≤c. Hence a+(b.c)≤c

Hence a+(b.c)≤(a+b).c →(1)

Therefore L is non modular if and only if the inequality (1) is strict for at least one triple (a,b,c) such that a≤c. When a=c, the two sides of (1) are equal by the absorption law.

L.H.S=a+(b.c)=a+b=b+a=a

R.H.S=(a+b).c=(a.c)+(b.c)=a+b=a

When a<c,suppose first that strict inequality holds in (1).

Put a'=a+(b.c) and c'=(a+b).c then by (1) a≤a'<c'≤c→(2) And b.c'= b.(a+b).c =b.c

a'+b= a+(b.c)+b=a+b

Moreover c'≤a+b, hence b+c'≤a+b≤b+c' by (1)

Therefore b+c'= a+b and dually a'.b=b.c

This shows that a' and c' have the common complement b in [b.c,a+b] and by (2) they are comparable but distinct.

Conversely if a',c' are distinct elements which are comparable and have a common complement in [u,v], say a'.b=c'.b=u , a'+b=c'+b=v and u$\leq$a'<c'$\leq$v then a'+(b.c')=a'<c'=(a'+b).c'.

Hence L is non modular.[3]

section 2.2.

Characterization of Modular lattices

There are two special lattices pentagon lattice and diamond lattice. M3 is modular. It is, however, not distributive.[3]

To see this,we have,

a$\cap$(b$\cup$c)=a$\cap$1=a and (a$\cap$b)$\cap$(a$\cap$c)=0$\cup$0=0.

Since a$\neq$0 ,M3 is not distributive. All lattices of four elements or less are modular. The smallest lattice which is not modular is the pentagon (N5).

We now discuss about the modular lattices and characterization theorems of modular lattices. In the definition of modular lattices, if c satisfies (a$\cap$b)$\leq$c$\leq$a, then we get that a$\cap$b$\cup$c=a$\cap$b$\cup$c=c.

The following theorem shows that to check modularity it is sufficient to consider c's that are in the interval [a$\cap$b,a].[3]

Theorem 2.2.1

A lattice L is modular iff ∀a,b∈L

d∈a∩b,a ⟹ a∩b∪d=d →(1) [3]

proof:

First, we show that (1) implies that L is modular.

Suppose c≤a but a∩b∩c is false.

We define d=c∪(a∩b) →(2). Clearly d∈[a∩b,a].

Consider a∩(b∪)d. Replacing the value of d from (2) we get, a∩(b∪(c∪(a∩b)))=c∪(a∩b)) → (3)

Now consider a∩(b∪c).

Using the fact that c≤c∪(a∩b) we get a∩(b∪c)≤a∩(b∪(c∪(a∩b))) → (4)

Combining (3) and (4), we get a∩(b∪c)≤c∪(a∩b).

Rewriting the equation gives us a∩(b∪c)≤(a∩b)∪c.

Since a∩(b∪c)≤(a∩b)∪c holds for all lattices (Lemma 2.1.2), we get, a∩(b∪c)=(a∩b)∪c.

Hence L is modular, which is what we wanted to show. Showing the other side, that modularity implies condition (1) is trivial.

The following lemma is useful in proving the next theorem.[3]

Lemma 2.2.2

For all lattices L,∀ a,b,c: let ν=a∩(b∪c) and u=(a∩b)∪c. Then,ν>u ⟹ [ν∩b=u∩b]∧[ν∪b=u∪b].[1]

proof:

We show the first conjunct. The proof for the second is similar.

$a \cap b = (a \cap b) \cap b \leq [(a \cap b) \cup c] \cap b = u \cap b$ (bydefinition of u)

$\leq \nu \cap b$ (since $\nu > u$)

$= (a \cap (b \cup c)) \cap b$ (by definition of ν)

$= a \cap b$ (since $b \leq (b \cup c)$).

We are now ready for another characterization of modular lattices.[1]

Theorem 2.2.3

A lattice L is modular if and only if it does not contain a sublattice isomorphic to N5.[1]

proof:

L violates the modularity if it contains N5 .

Now assume the contrary that L is not modular. Then, there exist a, b, c such that $a > c$ and $a \cap (b \cup c) > (a \cap b) \cup c$.

It can easily verified that $b \| a$ and $b \| c$. For example $b \leq a \implies (a \cap b) \cup c = b \cup c \leq a \cap (b \cup c)$. The other cases are similar. We let $\nu = a \cap (b \cup c)$ and $u = (a \cap b) \cup c$. We know that $\nu > u$. It is easy to verify that $b \| u$ and $b \| v$.

For example, $b < \nu \equiv b < a \cap (b \cup c) \implies b < a$.

From Lemma 2.2.2, $u \cup b = \nu \cup b$ and $u \cap b = \nu \cap b$ Thus $u, \nu, b, u \cup b$ and $u \cap b$ form N5.[1]

Theorem 2.2.4[Shearing identity][3]

A lattice L is modular if and only if $\forall x,y,z: x \cap (y \cup z) = x \cap ((y \cap (x \cup z)) \cup z$.

Proof:

Assume that L is modular.We have to prove that the shearing identity. We use the fact that $x \cup z \leq z$. $(y \cap (x \cup z)) \cup z = z \cup (y \cap (x \cup z))$ $= (z \cup y) \cap (x \cup z)$(by modularity)

$= (y \cup z) \cap (x cup z)$

Considering the right hand side of shearing identity.

$x \cap (y \cap (x \cup z)) \cup z$ $= x \cap ((y \cup z) \cap (x \cup z)$ $= x \cap (y \cup z)$ (since $x \leq x \cup z$).[3]

Chapter 3

Representation of modular lattice

.

section 3.1.

Definition:Representation

Let L be a modular lattices and V be a finite dimensional vector space over a field K. A representation of L in V is a morphism from L into the lattice L(V).Thus a representation:L→L(V) associates with each element x∈L a subspace ρ(X)⊆V such that for every x,y∈L, ρ(xy)= ρ(x)ρ(y)[4]

ρ(x+y)= ρ(x)+ρ(y)

Definition: Let ρ1and ρ2 be representations of a lattice L in vector spaces V1 and V2 respectively. We set ρ(x)=ρ1(x)⊕ρ2(x),∀x∈X, where ρ1(x) ρ2(x) is the subspace of V1⊕V2 consisting of all pairs (ε,η) such that ε∈ρ1(x) and η∈ρ2(x).

This defines a representation in the space V=V1⊕V2 .[4]

Definition: A representation ρ is decomposible if it is isomorphic to the direct sum set $\rho=\rho1\oplus\rho2$ of two nonzero representations $\rho1$and$\rho2$. The representation ρ in a vector space V is decomposible if and only if subspaces U1 and U2 such that U1.U2=0 and U1+U2=V and that ρ(a)=U1ρ(a)+U2ρ(a), $\forall$a$\in$L. [5]

Definition:Perfect lattice

An element a of a modular lattice L is called perfect if for every field K and every representation ρ:L$\rightarrow$L(V)=L(Kn) the subspace ρ(a)$\subseteq$V has the property that, there is a subspace U complementary to ρ(a) (i.e, Uρ(a)=0 and , U+ρ(a)=V) such that the subspace U and ρ(a) define a decomposition of ρ into the direct sum of sub representations. i.e, ρ(x)=Uρ(x)+ρ(a).ρ(x) for every element x$\in$L.[3]

Definition:Linear lattice

A modular lattice L is called linear if for any x,y$\in$L and every representation ρ:L$\rightarrow$L(Kn) we have, x=y $\Longleftrightarrow$ ρ(x) =ρ(y).[3]

Definition:Linear equivalence

Two elements a and b of a modular lattice L are called linearly equivalent if ρ(a) = ρ(b)in every representation ρ:L$\rightarrow$L(Kn) for any K and n. In this case we write a$\cong$b.[3]

Definition:Projective space

Let V be an (n+1)-dimensional vector space over F. The projective space P(F,n) is the geometry whose points, lines, planes,.... Are the vector subspaces of V of dimensions 1, 2, 3... A projective space of dimension n over a field F can be constructed by "projection" of an (n+1)-dimensional vector space over F. Note also that a d-dimensional projective space is a d+1-dimensional vector subspace.[3]

Definition:Completely irreducible

A representation of a modular lattice L in a space V over a field K of characteristic zero is called completely irreducible if ρ(L)≅P(Q,m).(i.e, projective space over the field Q of rational numbers). Here we denote by ρ(L)the sublattice of L(Kn) consisting of all elements a,a∈L.[3]

Case 1: First we construct the representation ρt,l in the vector spaces Vt,l ,t∈1,2,3,. . . ,l∈1,2,3,. . . .

Clearly a representation ρ of Dr in a space V is completely determined by the subspaces (ej),j=1,2,. . . of V. We set ρt,l(ej)=Ej,l⊆Vt,l.

Let l=1 then ρt,1;t∈1,2,3,. . . ,r is the representation in the one dimensional space V≅K1 such that ρt,1(ej)=0 if j≠t and ρt,1(ej)=V. Now let t¿1. We denote by Wt,l the linear vector space over K with the basis $\eta\alpha$ where α=(i1, i2,. . . . . . il-1,t) ranges over the whole set At(r,l)(α∈At(r,l) ⟺ α=(i1, i2,. . . . . . il-1,t)and t fixed). We denote by Zt,l the subspace of Wt,l spanned by all possible vectors. gα,k=σikηi1,i2,. . . .ik. . . ..il-1,t , where 1≤k≤ l-1 and the summation is over those α=(i1, i2,. . . . . . il-1,t)in which i1,i2,. . . .ik-1,ik+1,. . . il-1 are fixed. Next we set Vt,l =Wt,l/Zt,l. The images of the vectors $\eta\alpha$ in the factor space Vt,l are denoted by ∈α. Thus Vt,l is a vector space over K spanned by the vectors ∈α for which σik∈i1,i2,. . . .ik. . . ..il-1,t=0 ∀k.

By Ej,t we denote the subspace of Vt,l spanned by all vectors ∈α such that α=(i1, i2,. . . . . . il-1,t)=(j,i2,. . . . . . il-1,t) where i1=j is fixed. We define a representation ρt,l in Vt,l by setting ρt,l(ej)=Ej,t.

Case 2: For l=1, ρ0,1 is the representation in the one dimensional space V≅K1 such that ρ0,1(ej)=0 ∀ i=1,2,. . . r.

For l¿1, we define a set A0(r,l) in the following way, A0(r,l)=α=(i1, i2,. . . . . . il-1,0): iλ∈I=1,2,. . . ..r, i1≠i2,i2≠i3,. . . .il-2≠il-1. Clearly A0(r,l)≠A0(r,l-1).

The representation ρ0,1is constructed on A0(r,l). The vector space over K with the basis $\eta\alpha$,where α∈A0(r,l) is denoted by W0,l. Further let Z0,l be the subspace of W0,l spanned by all possible vectors gα,k=σikηi1,i2,. . . .ik. . . ..il-1,0. We denote by V0,l the factor space W0,l/Z0,l and by ∈α the image of $\eta\alpha$ under the canonical map W0,l→V0,l. The subspace of V0,l spanned by ∈α with α=(i1, i2,. . . . . . il-1,0)=(j,i2,. . . . . . il-1,0) is denoted by Ej,0. We define a representation ρ0,1 in V0,l by setting ρ0,l(ej)=Ej,0.[3]

Definition:Admissible[5]

Let ρ be a representation of modular lattice L in a linear space V. A subspace U of V admissible relative to if for any x,y∈L one of the following conditions is satisfied.

1. U(ρ(x)+ρ(y))=Uρ(x)+Uρ(y)

2. U+ρ(x)ρ(y)=(U+ρ(x))(U+ρ(y))

[5]

Proposition 3.1.1

Let ρ be a representation of a lattice L in space V. Let U be a subspace of V and let U"=V/U the factor space. Let θ:V→U"be the canonical mapping. Then the following conditions are equivalent.

1. The subspace U is admissible relative to ρ
2. The correspondence defines x$\rightarrow$Uρ(x), a representation in U.
3. The correspondence defines x$\rightarrow\theta\rho$(x) a representation in U".[5]

Proof: First we show that (i) $\Longrightarrow$ (ii).

Let U be an admissible subspace. Then Uρ(x+y)= U(ρ(x)+ρ(y))=Uρ(x)+Uρ(y)

Moreover Uρ(xy)=Uρ(x)ρ(y)=(Uρ(x).Uρ(y)) Consequently the rule x$\rightarrow$Uρ(x) defines a representation in U.

This representation is called admissible, it is also called the restriction of ρ to U and is denoted by ρ/U. In the similar way (ii) and (iii) can be proved.[5]

Proposition 3.1.2

A representation $\rho\in$R(L,K), the set of finite dimensional representations of L over K in V is decomposible if and only if there exist nonzero subspaces U1,U2,. . . ..Un such that

V=U1$\oplus$U2. . . ..$\oplus$Un and if $\forall$x$\in$L,

ρ(x)=σi=1n Uiρ(x)$\rightarrow$(1)[6]

Proof The necessity of (1) is clear. To prove the sufficiency we show that every subspace Ui is admissible.

i.e, Uj(ρ(x)+ρ(y)=Ujρ(x)+Ujρ(y) for any x,y$\in$L By (1), ρ(x)+ρ(y)=σi=1n Uiρ(x)+σi=1nU

Using Dedekind's axiom we find;

Uj(ρ(x)+ρ(y))=[Uj (σi=1n Uiρ(x)+σi=1n Uiρ(y)] =Uj(Ujρ(x)+Ujρ(y)+σi$\neq$jn (Uiρ(x)+Uiρ(y)) =Ujρ(x)+Ujρ(y)+Ujσi$\neq$jn(Uiρ(x)+Uiρ(y)) Note that Ujσi$\neq$jn(Uiρ(x)+Uiρ(y Ujσi$\neq$jnUi=0

Consequently Ujρ(x)+ρ(y)=Ujρ(x)+Ujρ(y)

This proves that every subspace Ujis admissible and means that the correspondence xUj xdefines a representation Ujin Uj. It is easy to check that=Uj ; where is over i=1,2,. . . .n and also so that is decomposible.

We now assume that V=U1$\oplus$U2. . . ..$\oplus$Un and that each of the subspaces Ui is admissible relative to ρ. The following example shows that we cannot in general assert that is equal to the sum of its restriction ρUj.[6]

Proposition 3.1.3[5]

Let ν+(l)$\in$B+(l)and ν+(m)$\in$B+(l). If l< m then ν+(l)$\supseteq$$\nu$+(m) Similarly if l< m then ν-(l)$\subseteq$$\nu$-(m) where ν-(i) $\in$B-(i) To prove this we need two lemma

Lemma 1[5]

Let α= (i1,i2, . . . ,il) $\in$ A(r,l) for l$\leq$2. We write

$\pi() = (i1, i2, ..., il - 1). Thentheelementse$ and e$\pi\alpha$ of Dr can be ordered as follows e$\pi$$\supseteq$e$\alpha$.

Proof: The proof is by induction on l [5].

Let l=2 i.e; α=(i1,i2).Then the elements eα =ei1i2=ei1Σji1i2 ej and eπ(α=eil. Clearly e$\alpha$$\subseteq$e.

Suppose that the lemma has been proved for every $\lambda$$\in$A(r,$\lambda$) with λ<1. We prove it for α = (i1,i2, . . . ,il).

By definition eα =ei1,i2,. . . ,il=Σ$\beta\in\tau(\alpha)$eβ;where $\tau(\alpha)$ = β=(k1,k2, . . . ,kl-1) ∈ A(r, l-1) / k1∉i1, i2, k2 ∉ i2, i3, . . . , kl-1 ∉il-1, il

Similarly e$\pi(\alpha$=ei1,i2,. . . ,il=Σ$\beta\in\tau(\pi(\alpha))e\beta$; where $\tau(\pi(\alpha))$= β' = (m1 ,m2,. . . ,ml-2)∈ A(r, l-2) /m1∉ i1, i2, m2∉ i2, i3, . . . , ml-2 ∉il-2, il-1

Clearly for any β=(k1,k2, . . . ,kl-1) ∈$\tau(\pi(\alpha)$ we can find an element β'∈$\tau(\pi(\alpha)$ such that β'=$\pi(\beta)$=(k1,k2, . . . ,kl-2).

By induction on such and β'=$\pi(\beta)$) we have eβ⊆e$\pi(\beta)$.

Consequently Σ$\beta\in\tau(\alpha)$eβ⊆Σβ'∈$\tau(\pi(\alpha))$eβ Hence eα⊆e$\pi(\alpha).\tau$.[5]

Lemma 2[5]

Let ht(l-1)∈B+(l-1) for l>1 and t ∈1.Then eα ⊆ht (l-1) for every α∈A(r,l).

Proof of proposition:[5]

The maximal element of B+(l) is VI,l=Σi∈l ei(l)=Σα∈A(r, l)eα and the minimal element of B+(l-1) is Vθ,l-1=∩t∈l ht . It is clear from lemma 2that for any α∈ A(r, l),∩t∈l ht(l-1)⊇eα

Therefore V,l-1= ∩t∈l ht(l-1)⊇Σα∈A(r, l)eα=VI,l.

Now if ν+(l-1) and ν-(l-1) are arbitrary elements of B+(l-1) B+(l) respectively then,ν+(l-1)⊇Vθ,l-1⊇ν+l.

The corresponding statements for the cubicles B-(l) and B-(m) is obtained by duality.

We denote by B+ the subset of Dr that s the union of the B+(l) ,l= 1,2,.. Similarly B-=∪l=1 B-(l).

Hence the proposition.

Chapter 4

Cubicles

.

In this section we deal with the representation of the modular lattice Dr with r generators e1,e2,. . . ,er. It is dealt with the construction of the sublattice B of Dr whose elements all are perfect.Here the lattice Dr is the r-dimensional lattice with countably many elements.

section 4.1.

Constructions

We construct sublattices B+(l) and B-(l) each consisting of 2r perfect elements. We call B+(l),the lth upper cubicle and B-(l),the lth lower cubicle.[4]

To define the upper cubicle B+(1) we set ht(1)=Σj≠t ej.Then the upper cubicle B+(1) is the sublattice of Dr generated by the elements h1(1), h2(1),. . . , hr(1).Thus B+(1) is isomorphic to the lattice of vertices of an r-dimensional cube with the natural ordering. The element hi(1) corresponds to the point (1,1,. . . ,1,0,1, . . .,1,1) with 0 in ith place. Now we construct the lattice B+(l) with l¿1 from certain important polynomials ei1,i2,..il. Now we proceed to define this polynomials. Let l$\leq$ and I=1,2,. . . ,r. We denote by A(r,l) the set whose elements are sequence of integers α= (i1,i2, . . . ,il) with i$\in$I such that i1i2,i2i3, . . . ,il-1$\neq$il. In particular A(r,1) = 1.

For fixed a$\in$A(r,l) we construct a set $\psi(\alpha)$ consisting of elements $\beta$$\in$A(r,l-1) in the following way ; $\psi(\alpha)$ = β=(k1,k2, . . . ,kl-1)$\in$A(r,l) / k1 $\notin$i1, i2, k2$\notin$ i2, i3, ... kl-1$\notin$ il-1, il Note that k1$\notin$k2, k2$\notin$k3, . . . ,kl-2$\notin$kl-1because βA(r,l-1).With each $\alpha$$\in$A(r,l) we now associate an element e$\alpha$$\in$Dr by the following rule; j$\neq$i1, i2 Let l=1 and α=(i1) we set eα=ei1

Forl=2 and α=(i1,i2) with i1$\neq$i2 we set eα=ei1i2=eil$\Sigma$$\beta$$\in$$\psi(\alpha)$ eβ=Σj$\neq$i1,i2 ej.

In general for arbitrary l and $\alpha$$\in$A(r,l) we set by induction eα = ei1,i2,..il = ei1$\Sigma$$\beta$$\in$$\psi(\alpha)$ eβ Now we introduce the elements ht(l). We denote by At(r, l) the subset of A(r, l) consisting of all α= (i1,i2, . . . ,il-1,t) whose last index is fixed and equal to 't'.

We set et(l) =$\Sigma$$\alpha$$\in$At(r,l) e$\alpha$ and ht(l)=Σj$\neq$1ej(l).

Thus we define the sublattice of Dr generated by the elements h1 (l), h2(l),. . . ,hr(l) as follows.

Definition: The subsets of Dr generated by h1 (l), h2(l),. . . ,hr(l) is called the lth upper cubicle B+(l). The collection of elements of all of the cubicles B+(l) is also a lattice which is denoted by B+. We denote by B-(l) the sublattice of Dr dual to B+(l).The sublattice B+(l) is called the lth lower cubicle.[4]

Atomic representation and their connection with representation of B+(l) [5]

We define the most trivial among the indecomposable representation of Dr the atomic representations of Dr the atomic representation for ρj,1for j$\in$1,2,. . . r.

a. The representation ρ0,1 n the one dimensional space V$\cong$K1is defined by ρ0,1(ei) = 0 $\forall$ i$\in$1,2,. . . ,r. It follows that ρ0,1(x) = 0 $\forall$ x$\in$ Dr.

b. The representation ρt,1 for t$\in$1,2,. . . r in V$\cong$ K1 is defined by ρt,1(ei) = 0 for t=j and ρt,1(et)=V.

We have B+(l) is a Boolean Algebra with minimal element Vθ,l=$\cap$t$\in$l ht(l). Now we prove that Vθ,l is perfect that is the restriction of ρ to ρ(Vθ,l) s a direct summand of ρ . We denote ρ(Vθ,l)by Vθ,l.

Proposition 4.1.1[6]

Let ρ be a representation of Dr in a space V over K and let Vθ,l be the ρ=($\oplus$j$\in\cup$0ρJ,l)$\oplus$ $\tau\theta$, 1 where $\tau\theta$, 1 is the restriction of ρ to the subspace Vθ,l= ρ(Vθ,l) and each ρj, l is a multiple of the atomic representation ρj,l. i.e; ρJ,l=ρj,l$\oplus\rho$j,l$\oplus$. . . $\oplus$j,l(m times) where mj$\geq$0.

Proof:[6]

We choose Uj such that V=($\oplus$j$\in\cup$0 Uj)$\oplus$Vθ,l where 0=$\cup$0=0,1,,r and ρ decomposes into a direct sum relative to these subspaces.

We claim that Uj can be chosen as subspaces satisfying the following relations

U0Σi=1r ρ(ei)=0, U0+Σi=1r ρ(ei)= v $\rightarrow$(1)

And for any j≠0 , j∈

Ujρ(hj) =0 ,Uj+ ρ(ejhj) =ρ(ej); where hj=hj()=Σt≠j et → (2)

Step 1

Any element of B+(l) can be written in the form;

Va,l=Σ i∈a ei+Σ j∈a' ejhj → (3) where; a is a subset of =1,2,,r and a'=-a.

We write Va,l=ρ(Va,l) and claim that if in V subspaces Uj, j=0,1,. . . ,r are chosen to satisfy (1) (2) then for any a⊆, Va,l=Σj∈a Uj +Vθ,l → (4)

We prove (4) first for the case of one element subsets a =t, t∈1,2,. . . ,r.i.e; prove that Vt,l=Ut+Vθ,l →(5) Let ρ(ei)=Ei and ρ(hi) = Hi

Then from (3) Vt,l=Σi≠t EiHi and Vθ,l=Σi=1r EiHi.

This proves (4).Then we find that Ut+Vθ,l= Ut+Σi=1r EiHi =Ut+EtHt+Σi≠t EiHi

From (2) Ut+Et Ht=Et,consequently Ut+Vθ,l= Ut+Σi≠t EiHi=Vt,l

This proves (5)

We have B+(l) s a Boolean Algebra and that Va∪b,l=Va,l+Vb,l for any subset a,b⊆. n particular Va,l=Σt∈a Vt,l.

Thus every subspace ρ(Va,l)=Va,l can be represented as a sum Va,l=Σt∈a Vt,l. Putting Vt,l=Ut+Vθ,l in this we obtain

Va,l=Σt∈a (Ut+Vθ,l) =Σt∈a Ut+Vθ,l.

This proves (4).

Step 2

We show that our chosen subspaces Uj are such that

V≅Vθ,l⊕U0⊕U1⊕. . . Ur.

We write at=t+1,,r and Wt=Vat,l=ρ(Vat)

Wt=Σi=1t EiHi+Σj=t+1r Ej

t follows easily from the relations EiHi⊆Ei that the subspaces Wt,t∈1,2,,r form a chain, Wr⊆Wr-1⊆. . . ⊆W2⊆W1⊆W0⊆V,where Wr=Σi=1t EiHi=Vθ,l, W0=Σ i=1r Ei =V,l.

We claim that Uj,j∈0,1,2,,r subject to (1) and (2) are connected with the Wj by the following equatons,

W0+U0=V and W0U0= 0 → (6)

And for every t∈1,2,,r, Wt+Ut= Wt-1 →(7)

UtWt=0 → (8)

Note that (6) is the same as (1) because ,W0=Σi=1r Ei+Σi=1rρ (ei)

We have proved earlier that Va,l=Σi∈a Ui +Vθ,l∀a⊆.

Consequently Wt=Vat,l=Σj=t+1r Uj+Vθ,l=Σj=t+1r Uj+Wr

Now (7) evidently follows from this equation. Note that

Wt=Σi=1t EiHi +Σj=t+1r Ej⊆EtHt+Σj≠t Ej=EtHt+Ht=Ht.

From this,using (2)(UtHt=0) we obtain UtWt⊆HtUt=0. i.e; UtWt=0.

This proves (8).

It follows easily from (6) - (8) that V=Σt=0r Ut+Wt=Σt=0r Ut+Vθ,l and that this sum is direct.

Step 3

We claim that for every i∈1,2,. . . ,r , Ei=Σj=0r EjUj+ EiVθ,l → (9)

To prove this we first show that EjUj=0 for i$\neq$j and EiUi=Ui.

For by construction U0Σi=1r Ei=0 and UjHj=UjΣi$\neq$j Ei=0, Ui+EiHi=Ei.

Consequently,∀i$\neq$0 we have

1. U0Ei=0

2. UjEi=0,for i$\neq$0,j$\neq$1

3. EiUi=Ui

Thus we can rewrite the right hand side of (9) in the following form;

Σi=1r EiUj +EiVθ,l

Let us find EiVθ,l

By definition Vθ,l= ρ(Vθ,l)=ρ(∩j=1r hj)=∩j=1rHj Where Hj=Σt$\neq$j Et.Hee EiHj=Ei,where i$\neq$j and therefore

EiVθ,l=Ei∩j=1r Hj=∩j=1r EiHj=EiHi which is true by construction (2).

This proves (9).

Step 4

We combine the results of steps 2 and 3 we have proved that V=Σi=1r Ui+Vθ,l and that this sum is direct. Further we have proved that every subspace Ei(c=ρ(Ei)) is represented as sum,Ei=Σj=0r EiUj+EiVθ,l. ρ splits into the direct sum

ρ=(⊕j∈0ρJ,l) ⊕ $\tau\theta$, 1 where ρJ,l=ρUj and $\tau\theta$,l=ρVθ,l

Step 5

We claim that ρJ,l is a multiple of the atomic representation ρJ,l.

i.e;ρJ,l=ρj,l⊕j,l⊕. . . ⊕ρj,l.

First we study the representation ρ0,l . The subspace U0 is admissible,therefore ρ0,l(ei). We have proved that U0Ei=0 ∀i. Thus the sub representation ρ0,l in U0 is such that ρ0,l(ei)=0 ∀i.

If dim U0=m0>0, then ρ0,l is different from zero and clearly splits into the direct sum ρ0,l=ρ0,l⊕ρo,l⊕. . . ⊕0,l of atomic representation ρ0,l.

Similarly for the ρJ,l,with j≠0, we obtain

ρj,l(ei)=Ujρ(ei)=EiUj=0, if i≠j= Uj,if i=j

If dim Uj=mj> 0 then it is easy to see that ρj,l splits into a direct sum of atomic representations ρj,l=ρj,l⊕ρj,l⊕. . . ⊕ρj,l.[5]

Lemma 4.1.1[3]

Let L be an arbitrary modular lattice and e1, . . . ,er a finite set of elements of L. Then the sublattice B generated by the elements hj=i≠jei , j = 1,2,. . . ,r is a Boolean Algebra.

Proof: Let C be a non empty subset of I= 1,2,. . . ,r. We claim that the following identity holds in L ; ∩i∈C hi =Σi∈C eihi +Σk∈(I-C) ek→(1)

If C consists of a single element C = j then (1) takes the form hj= ejhj+Σk≠j ek→ (2)

By definition Σk≠j ek=hj. Thus in the case C= j we must prove that hj=ejhj+C1 which is obviously true.

Suppose that (1) is proved for every subset C of m elements (m<r). Now we show that (1) holds for every subset C1 containing C and consisting of m+1 elements. Suppose for example that C1 = C∪s where s∉C.

Then; $\cap j\in C1$ hj= hs($\cap j\in$Chj)

= $\Sigma t\neq s$ et ($\Sigma i\in C$ eihi +k$\in$(I-C)ek) =$\Sigma t\in s$ et($\Sigma i\in C$ eihi+$\Sigma k\in$(I-C1)ek+es)

It follows from s$\notin$C that; $\Sigma t\notin s$ et$\supseteq$ $\Sigma i\in C$ ei$\supseteq$ $\Sigma i\in C$ eihi and

$\Sigma t\neq s$ et $\supseteq\Sigma k\in$(I-C) ek

Then by Dedekind's axiom

$\cap j\in C1$ hi =$\Sigma i\in C$ eihi +$\Sigma k\in$(I-C) ek+($\Sigma t\neq$set)es

= $\Sigma i\in C$ eihi +$\Sigma k\in$(I-C)ek +es hs =$\Sigma i\in C$ eihi +$\Sigma k\in$(I-C)ek

We have denoted by B the sublattice of L generated by the elements hj=$\Sigma t\neq j$ et. We claim that every element v$\in$ B can be written in the form v=$\Sigma i\in a$ ei+$\Sigma j\in a'$ ejhj where $\subseteq$a and a'= I-a.

Note that in the case v=hj we have proved in (2) that hj=$\Sigma i\in$I-a ei+ejhj Now let v1v2 be two elements of L such that; vq=$\Sigma i\in aq$ ei+$\Sigma j\in a'q$ ejhj (q=1,2)

It easily follows from the identity eih+ei that v1+v2=$\Sigma i\in$(a1$\cup$a2)ei+$\Sigma j\in$(a1$\cup$a2)ejhj

Applying (1) we can write in the case a$\neq$I;

$\Sigma i\in a$ ei+$\Sigma j\in a'$ ejhj =$\cap j\in a$ hj

In accordance with this identity, in case aq$\neq$I we have;

v1v2=($\cap i\in a1$ hi)($\cap i\in a2$ hi)=$\cap i\in$(a1'$\cup$a2')hi

Since we have a1'$\cup$a2' = (a1$\cap$a2)' we have;

v1v2=$\cap i\in$(a1$\in$a2)'hi =$\Sigma i\in$(a1$\cap$a2)ei + $\Sigma j\in$(a1$\cap$a2)ejhj

It remains true when a1= I or a2 = I. We proved that every element v$\in$B can be written in the form v =$\Sigma i\in a$ ei+$\Sigma j\in a'$ ejhj.

We denote such an element by va. The set of all subsets of I = 1,2,. . . ,r is denoted by B(I).We know that B(I) is a Boolean algebra with 2r elements. We have proved that νa+νb=νa$\cup$$\nu$b and νa.νb=νa$\cap$$\nu$b for an a,b$\in$B(I).This shows that the correspondence a $\rightarrow$$\nu$a is a morphism of B(I) onto B.

Hence B is a Boolean algebra with 2m elements where m$\leq$r.[3]

Corollary 4.1.2[3]

Each sublattice B+(l),B+(l) ; l= 1,2, . . . of Dr s a Boolean algebra.

CONCLUSION

This paper presented the very basic concept about lattices and modular lattices especially the representation of modular lattice Dr. The fundamental concepts, properties and propositions about lattices were discussed thoroughly. The different kinds of lattices with examples and its properties were taken in particular in first chapter. Modular lattice and its characterisation is considered in the second chapter. Representation of modular lattices are considered and defined in the third chapter. Finally the sublattices of modular lattice named cubicles were introduced and hence the upper cubicle and the lower cubicle were defined. The construction of sublattices B+(l) and B-l were discussed more clearly through the fourth chapter.

Apart from this, the theory of posets and lattices has many practical application in distributed computing. We believe that the future will bring even more applications of the theory of order to distributed computing. For example, the concepts of Mobious function, Zeta polynomial and Generating functions in a posets or modular lattices,geometric lattices etc.

Bibliography

[1] G.Birkhoff.Lattice theory.Providence,R.I.,1967,third edition.

[2] R.P.Dilworth.A decomposition theorem for partially ordered sets.Ann.Math.51,pages 161-166,1950

[3] B.A.Davey and H.A.Priestley.Introduction to lattices and order.Cambridge university press,Cambridge,UK,1990

[4] E.Egervary.On combinatorial properties of matrices.Mat.Lapok,38:16-28,1931

[5] C.J.Fidge.Partial orders for parallel debugging.Proc.of the ACM SIGPLAN/SIGOPS.Workshop on parallel and Distributed Debugging,(ACM SIGPLAN Notices),24(1):183-194,January 1989.

[6] V.K.Garg.Principles of Distributed systems.Kluwer Academic Publishers,Boston,MA,1996