A STUDY ON GRACEFUL GRAPH

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DISSERTATION SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF BACHELOR'S DEGREE IN MATHEMATICS

JOINTLY

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Affiliated to Mahatma Gandhi University, Kottayam

DECLARATION

We hereby declare that this project entitled A STUDY ON GRACEFUL GRAPHS is a bonafide record of work done by us under the supervision of Prof.Disna Sharon, and the work has not previously formed the basis for any other qualification, fellowship, or other similar title of other university or board.

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This is to certify that Dissertation entitled **A STUDY ON GRACEFUL GRAPHS** submitted jointly by Miss. Diya Thaquvi Noushad, Mr. Muhammed Anas M.A, Miss. Aswathy P Biju, Miss. Krishna M S ,Miss. Navya Sunilkumar, in partial fulfillment of the requirements for the BSc. Degree in Mathematics, is a bonafide record of the studies undertaken by them under my supervision at the Department of Mathematics, Bharata Mata College, Thrikkakara during the academic year 2020-23, This dissertation has not been submitted for any other degree elsewhere.

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INTRODUCTION

The theory of labelling is just one of the many subjects that can be covered by graph theory .The term " β-labelling" refers to a new method of graph labelling developed by A. Rosa in 1966. In this method, each edge in the graph is given a unique label based on the absolute difference between the labels of its end vertices labelled from 0 to n. A few years later, S.W. Golomb gave the practise the modern name graceful labelling.

Consider the scenario where we want to break down a complete graph G into isomorphic trees. In other words, we wish to partition the edges of G so that each set of the resulting subgraphs is isomorphic to a certain tree T.Ringel postulated that the complete graph K_{2m-1} can be broken down into 2m-1 trees isomorphic to T for any tree T with m vertices. Rosa was able to demonstrate that the Ringels conjucture holds if every tree enables a graceful labelling using the definition above.

The graceful graph developed its own worth after years of research. The "Open Problem of the Month" in Davis S. Johnson's NP-completeness column from 1983 is the decision problem of gracefull labelling. In addition, one of the key subjects of the International Workshop on Graph Labelling is graceful labelling.

The topic of graceful labelling on a graph has been studied over the past 50 years, and there are still many properties to be discovered.Here, we provide a succinct analysis of gracious labelling.

CHAPTER-1

PRELIMINARIES

In this section ,we give the majority of the definitions and notations for graph theory used in graceful labelling

A graph G is an ordered pair (V,E), where V is a collection of elements known as vertices and E is a collection of unordered pairs of vertices from V called edges that are distinct from one another. If two vertices u and v are joined by an edge, then we say that they are neighbouring. This is indicated by the notation e = uv. The set of neighbours of a vertex u is also known as the set of neighbouring vertices of u and is denoted as N(u). The number of a vertex's neighbours, or |N(u)|, determines the vertex's degree, or d(u). i.e., d(u) = |N(u)|

When the vertex set and the edge set for a given graph G are not explicitly supplied, they are referred to as V(G) and E(G), and the numbers m and n are used to denote the number of vertices and edges, respectively.

A graph that has V(H) ⊆ V(G) and E(H) ⊆ E(G) is said to be a subgraph of G. The subgraph of G caused by W, designated as G[W], is the graph H = (W, F) such that, for any u, v Є W, given a graph G = (V,E) and a subset WCV. If uv Є E, uv Є F follows. We claim that G's induced subgraph H. Subgraphs and induced subgraphs can be defined equivalently in terms of vertices and edges being deleted: If H is obtained by deleting G, then H is an induced subgraph of G.

H is a subgraph of G that is induced, as we say. Subgraphs and induced subgraphs can be defined We claim that G's induced subgraph H. Subgraphs and induced subgraphs can be defined similarly in terms of vertices and edges being deleted: H is an induced subgraph of G if vertices are deleted, and H is a subgraph of G if vertices and edges are deleted.

if we take a graph, a walk is a finite sequence W = (V_0, V_1, ... V_k) of vertices where the edges are V_i V_{i+1}. W is said to be a trail, if the walk W doesnt travel through the edge twice and W is said to be a path, if it does not travel through the vertex twice. A path with initial vertex u and terminal vertex v is called a uv path

The distance between two vertices, u and v, is the length of the shortest path between them and is denoted as dist(u, v). The length of a path is equal to the number of its edges. Dist(u, v) = ∞. if there is no path between u and v.

If the first and last vertices are the same, a walk is said to be closed. cycle is a closed trail with distinct vertices at every point until the last.

A graph's Eulerian trail, also known as an Eulerian path, is a path that precisely travels each edge of the graph once . Similar to this, an Eulerian tour (also known as an Eulerian cycle) is a cycle that travels exactly once along each edge. If a graph allows for an Eulerian cycle, it is Eulerian.

Every pair of vertices in a graph G is referred to as being connected if a path connects them. G is considered to be a tree if there is only one path that connects every pair of vertices. A linked graph with m - 1 edges is what a tree is equivalent to.

a graph of paths P_m is a connected graph with m vertices in which each verticel has a degree of no more than two. Graph of a cycle Every vertex in the connected graph C_m has a degree of 2, and it has m vertices.

A graph K_m is a complete graph with m vertices where each vertex is next to every other vertex. In contrast, an independent set is a group of graph vertices where no two vertices are next to one another. For an independent set with m vertices, we define I_m.

A graph with two sides also called bipartite graph A graph with the formula G = (V,E) exists if and only if P = (A,B) of V can be partitioned so that every edge of G connects a vertex in A to a vertex in B. Equivalently. If A and B are separate sets, then G is said to be bipartite. Another way to represent the bipartite graph is as G = (A,B,E).

The graph G = (V,E) is the join of two disjoint vertex set graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$. It has the properties that $V= V_1$ U 2 and $E = E_1 \cup E_2 \cup (uv: u \in V_1, v \in V_2)$, meaning that G is created by connecting each vertex of G_1 to each vertex of G_2.

The vertex labelling (or vertex colouring) of a given graph G = (V,E) is a function $f: V \rightarrow N$, while the edge labelling (or edge colouring) of a given graph G = (E,V) is a function $g: E \rightarrow N$. Intuitively, we are labelling the edges and/or vertices of the graph with different colours. The codomains are treated as a finite subset of N throughout this text, and we denote

$$[a, b] = \{a, a +1,....b\}.$$

Finding a vertex or edge labelling for a graph satisfying specific requirements comprises several graph theory problems. For instance, a proper vertex colouring is one in which neighbouring vertices have distinct colours, and it is a fairly well-known problem to determine the minimal k such that a proper vertex colouring f of G exists and | Im(f)|= k for a given graph G. In this situation, graceful labelling is what we're after.

CHAPTER-2

GRACEFUL LABELING

A graph G is gracefully labelled if the vertex labelling, f: V → [0,n], and the edge labelling, f_γ: E → [1,n], defined by f_γ: (uv) = |f(u)- f(v)|, are both injective. We define a graph G as being graceful if it allows for graceful labelling.

We just need to look at simple graphs or graphs without loops or parallel edges while considering graceful labelling. In a labelled graph, a loop would assume an edge label of 0, but in a graph with graceful labelling, G, the resulting edge label must be distinct and accept values of 1, 2, 3,...,n where n is the number of edges in G. In a labelled graph, parallel edges between a specific set of vertices would always assume the same edge label, violating the requirement that the edge label in a graceful labelling be distinct.

Although graceful labelling has been researched for 50 years, not many general conclusions have been reached .Since it is sufficient to display a graceful labelling for each graph in the class, the majority of the results focus on expressing the gracefulness of a graph class. On the other hand, results regarding a graph's lack of grace primarily rely on a necessary condition that is only true for Eulerian graphs or on attempts to label the graph gracefully up until a contradiction, which is generally ineffective.

Figure 2.1 graceful labelling of P_3 and $K_{1\,3}$

Let's label a path graph to get a better understanding of how to gracefully label a graph. Take a path graph P, and then allow

The set of vertices such that u k - 1 u k E(P m) for 0 k m is known as V (P_m)= {u_0, u_1,..., u_{m-1} }. We must name the vertices with integers from 0 to m - 1 so that each number in [1, m - 1] appears as an edge label because P has n = m - 1 edges. Since there is only one method to obtain an absolute difference equal to m - 1, which is to have a vertex with label 0 close to a vertex with label m - 1, we begin with edge label m - 1.Try labelling u 0 with 0 and u 1 with m - 1 as a result. Let's try to obtain an edge label with a value of m - 2 next. The only two approaches to obtain m - 2 as an absolute difference are as follows

m - 2 = |(m - 2) - 0| = |(m - 1) - 1| We can only obtain the edge label m - 2 by labelling u 2 with 1 because there are no more unlabelled adjacent vertices to u 0. If we continue using this method, our labelling will be as follows:

$$\mathbf{f(u_k)} = \begin{cases} \frac{k}{2} & , if\ k\ is\ even \\ m - \frac{k+1}{2} & , if\ k\ is\ odd \end{cases}$$

Now, all that is needed to demonstrate that f is really a graceful labelling of P m is for the edge label to materialise, which is anticipated to do so on the final edge.

$u_{m-2}u_{m-1}$. If n is even, then $f(u_{m-1}) = \frac{m}{2}$ and $f(u_{m-2}) = \frac{m-2}{2}$.

Consequently, $f_\gamma (u_{m-1}\, u_{m-2}) = \frac{m}{2} - \frac{m-2}{2} = 1$

An analogous argument provides the edge label 1 if n is odd. Consequently, the following premise is valid.

Statement 1: For any m ≥ 1, the path graph P_m is graceful.

In the second case, we look for a kind way to label all of the graph K_m. K_1 and K_2 are graceful because they are path graphs as well. For, K_3 and K_4

Figure 2.2 presents a graceful labeling for each one.

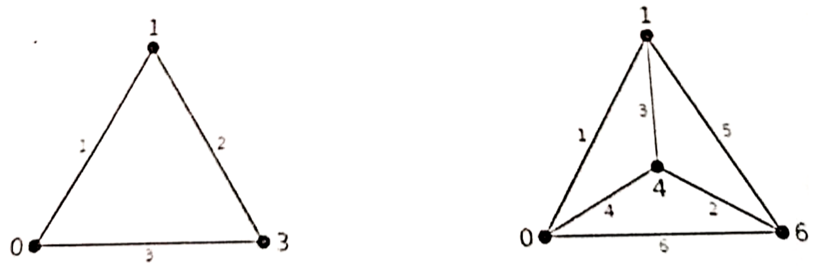


Figure 2.2 : Graceful Labeling of K_3 and K_4

Let's propose a characteristic of graceful labelings first before we analyse the general situation. If we swap every vertex label from k to n - k in a graph with graceful labelling, the resulting labelling is also graceful because the edge labels remain the same: the labels a and b on an edge's end vertices change to n – a and |a-b| = |(n-a)-(n-b)| and n - b. The complementarity property refers to this

As before, we now need a vertex with label 0 to be next to a vertex with label n in order to obtain the edge label n for K_m with m > 4. However, in this instance, each vertex is next to every other vertex. We can therefore name each vertex with 0 and without losing generality, any other one with m.

There are two ways to obtain the edge label n-1 : n-1 = |(n-1)-0| = |n-1|. The complementarity quality, however, enables us to select either without losing the generality. We obtain edge labels 1 and n-1 when we choose to label a vertex with 1. The edge label n-2 = |(n-2)-0| = |(n-1)-1| = |n-2| must now be obtained .Because it would result in a duplicate edge label, we cannot label a vertex with n-1 or 2. Therefore, the only choice we have is to label a vertex with n-2 to get the edge.2, n-2, and n-3 labels.

Since n-3 has already occurred on an edge, we must retrieve the edge label as the following edge.- n-4 = |(n-4)- 0| = |(n-3)-1|= |(n-2)-2| = |(n-1)-3| = |n-4|.. Again, there is only one way to label a vertex without producing duplicate edge labels, and that is to label it with 4, which yields edge labels 3, 4, n-6, and n-4. Five vertices have now been assigned labels. We would have n-6 = 4 as a duplicate edge label for K_5, though. The following edge label to obtain for m ≥ 6 is n - 5. However, each of the five approaches to obtain n-5 results in a duplicate edge label. As a result, the following premise is true and there is no method to get label n-5 on an edge

Statement 2: If and only if m ≥ 4, the complete graph K_m is graceful.

2.1BASIC DEFINITIONS

A complete graph with n vertices, a cycle of length n, and a tree, respectively, are denoted by K_m, C_m and Tin Figure 5. The Petersen graph is gracefully labelled as K_3, C_4, tree T in Figure 5:

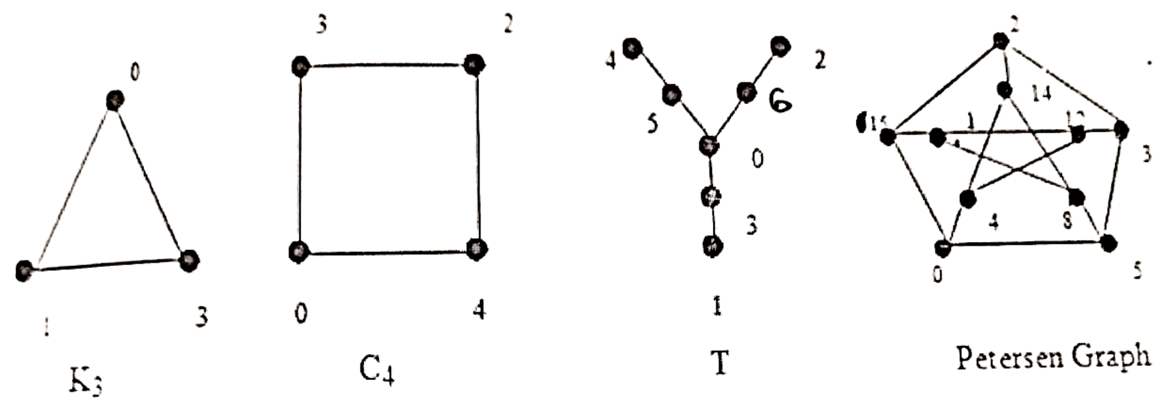


Figure 2.3 : Some Graceful Graphs

Not all graphs are graceful; for m > 4, C_5 and K_m are two examples. As seen in Figure 6, a given graph may have numerous different graceful labelings:

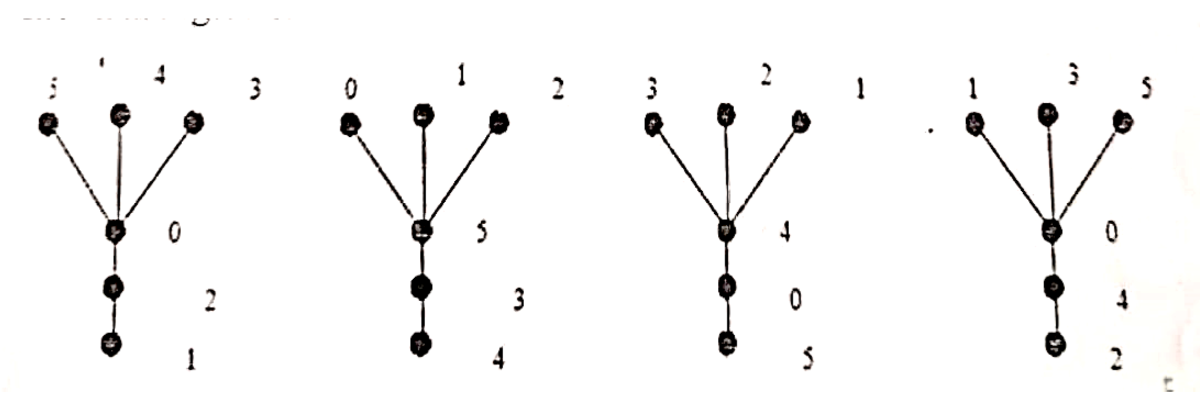
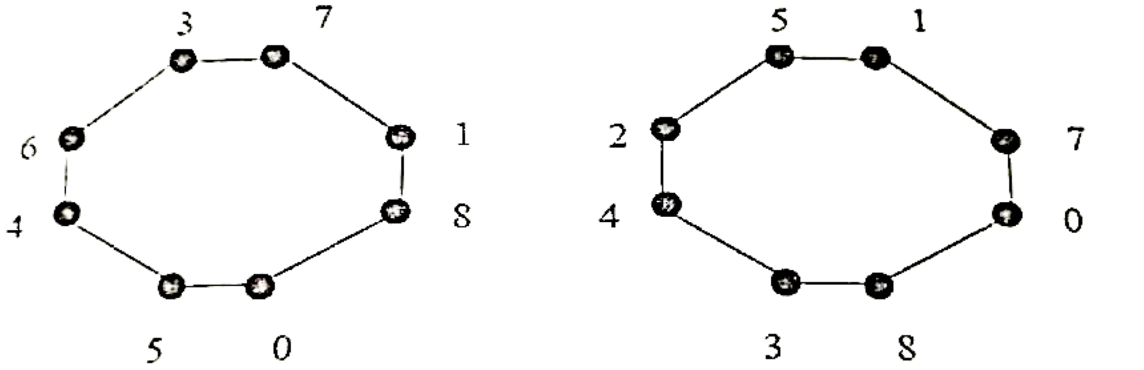


Figure 2.4 : Several Graceful Labelling of a Graph

There exactly are n! Graphs having n edges that are gracefully labelled. Additionally, a graceful graph's subgraph need not also be graceful. For instance, C_5 is a subgraph of the Peterson graph, but unlike Peterson, C_5 is not graceful.

Explanation 1: If a graph G = (V,E) has n = |E| edges, then if is a graceful labelling of G, then the valuation, $\psi^{\oplus}$ defined by $\psi^{\oplus}(v) = n - \psi(v)$ for all v Є V(G) is also a graceful labelling of G and is referred to as complimentary labelling (or complementary valuation) to ψ.



Ψ: α-labeling of C_8 $\quad$ $\Psi^{\ominus}$: complementary labeling of C_8

Figure 2.5 : Graceful Labelling and its Complimentary.

Explanation 2: A k-graceful labelling of a graph G = (V,E) with n = E(G) edges is a one-to-one mapping f of the vertex set V(G) into the set {0,1,2,...,n+k-1} such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is {k,k+1,k+ 2,...,n+k-1}.

Explanation 3: A wheel W_m is a network that is created from a cycle C_m by adding a new vertex and edges connecting it to all of the cycle's vertices; m is considered to be at least three. In Figure, a 7-graceful labeling of C_{15} and a 3-graceful labeling of W_7 are shown

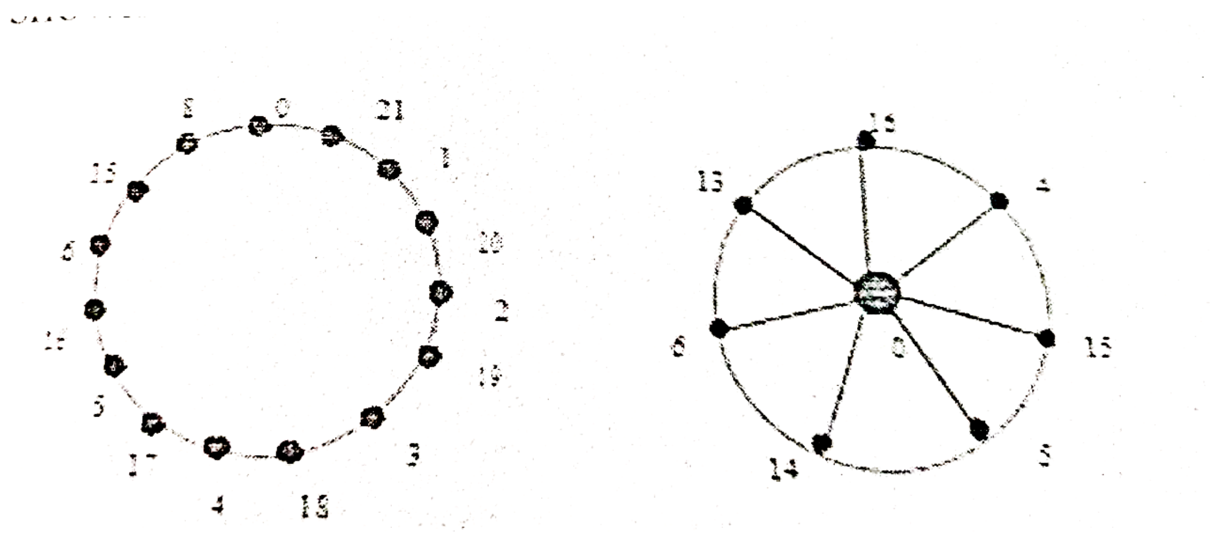


Figure2.6 :7-Graceful labeling of C_{15} and 3-graceful of W_7

It is clear that a 1-graceful labelling matches to the concept of graceful labelling that is generally used. If the graph G = (V,E) has a graceful labelling, then the graph G is k-graceful with the labelling ψ given as follows:

$\Phi(v) = (\psi(v) \text{ if } \psi(v) \leq \gamma,\ \psi(v) + k - 1 \text{ if } \psi(v) > \gamma, v \in V(G)),$

where ψ(v) refers to the vertex label of vertex v.

In Figure, an graceful labeling of C_4 changed to a 6-graceful labeling by using the above transformation

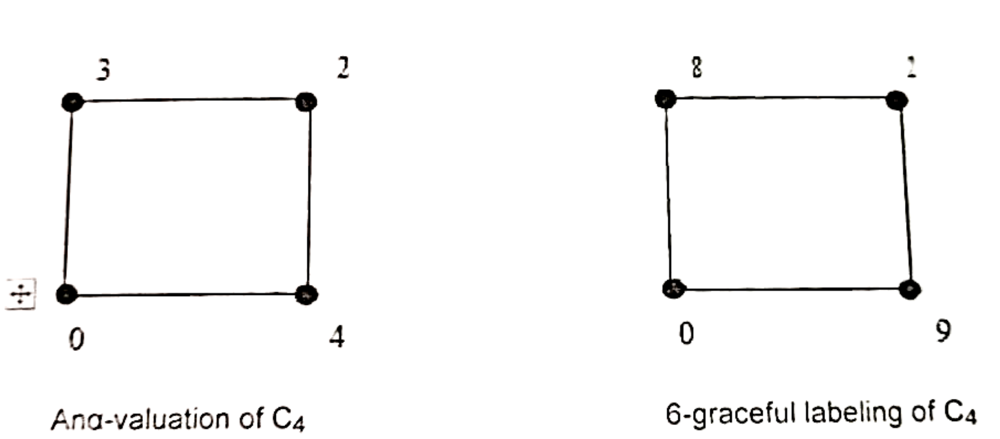


Figure 2.7: Transformation of an graceful labeling to k-graceful labeling for C_4.

Graphs that are arbitrarily graceful are sometimes described as being k-graceful for all k.

Explanation 4: A k-sequential labelling of a graph G = (V,E) with n = E(G) edges and m = V(G) vertices is a one-to-one function Γ from V(G) U E(G) to {k, k+1, k +2,...,n+m+k-1} such that for each edge e= {u, v} Є E(G), one has Γ(e) = Γ(u) - Γ(v).

A "k-sequential graph" is a graph G that allows k-sequential labelling. G is referred to as a sequential graph if it is an al-sequential graph. In Figure, the wheel W_6 and the cycle C_4 only sequential, although graph G_1 is 4-sequential:

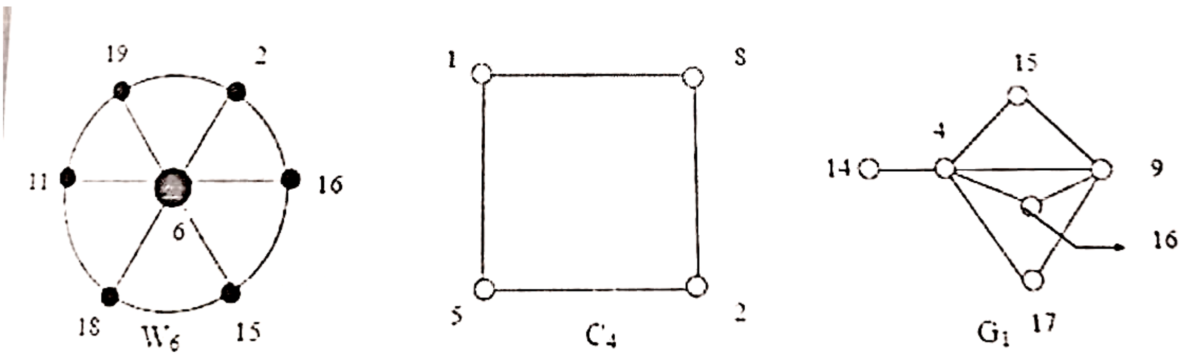


Figure 2.8: Examples of simply sequential graphs and a 4-sequential graph

We shouldn't be shocked if there is a relationship between graceful graphs and sequential graphs when we take into account how similar their concepts are. Let's define the following operation in two graphs before we describe the relationship between these two types of labelling:

Explanation 5: The join of the two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2E_2)$ denoted by $G_1 + G_2$, is defined as $V(G_1 + G_2) = V_1 \cup V_2$; $V_1 \cap V_2 = \emptyset$ and $E(G_1 + G_2) = E_1 \cup E_2 \cup I$ where $I = \{(v_1, v_2): v_1 \in V_1, v_2 \in V_2\}$. Thus Iconsists of edges which join every vertex of G_1 to every vertex of G_2.

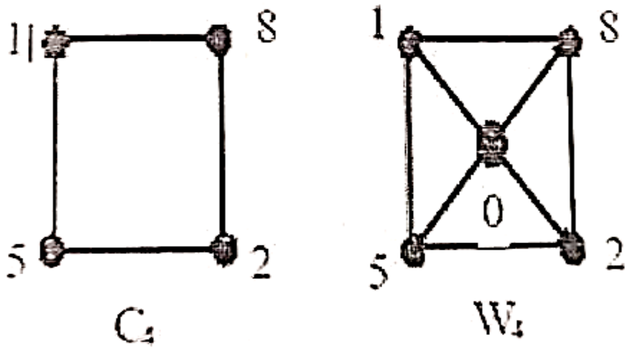


Figure 2.9: Corresponding 1-sequential labeling of $C_4$and graceful labeling of W_4

In Figure 2.9, we see that C_4 is simply sequential, then C_{4+V} , where v is the isolated vertex or in the other words W_4 has a graceful labeling ψ with $\psi(v) = 0$ as illustrated in Figure 2.8.

If and only if G + v has a k-graceful labelling ψ with $\psi(v) = 0$, then graph G is k-sequential.

For example in Figure 2.10, a 3-graceful labeling W_7, shown before in Figure 2.5, is transformed to a 3-sequential labeling of C_7 .

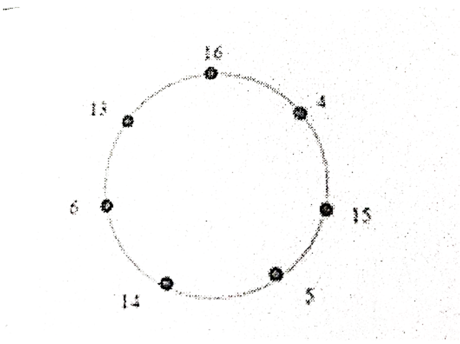


Figure 2.10: 3-sequential labeling of C_7

In the last ten years, the approaches for labelling a graph have expanded quickly. Numerous novel labelling techniques, including cordial labelling, harmonious labelling, elegant labelling, prime labelling, and sum labelling, have been researched.

CHAPTER 3

COMPLETE GRACEFUL GRAPHS

The complete bipartite graph $K_{a,b}$ is the graph that results from joining each of the "a" vertices with each of the "b" vertices in every way conceivable. It has m = a + b vertices and n = a x b edges. For all n_1 and n_2, the entire bipartite graph $K_{n1\ n2}$ has a -value.

The graceful labeling of $K_{3,3}$ is shown in Figure

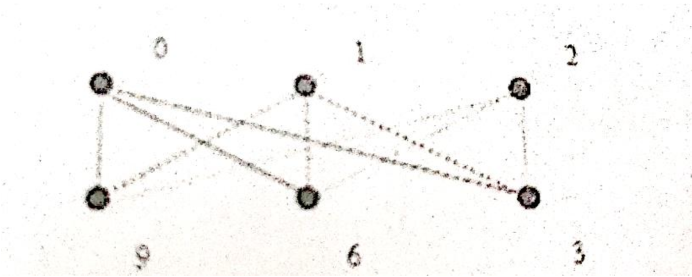


Figure 3.1: A Graceful labeling of $K_{3,3}$

Since $K_{n1,n2}$, has an α -labeling it is k-graceful too.

Star is the name given to the graph $K_{1,n}$. Only when k divides n is the star $K_{1,n}$ considered to be k-sequential.

$K_{n,n}$ is n-sequential for all n ≥ 1

In Figure we have shown the 3-sequential labeling for a star $K_{1,6}$ and a bipartite complete graph K 3,3.

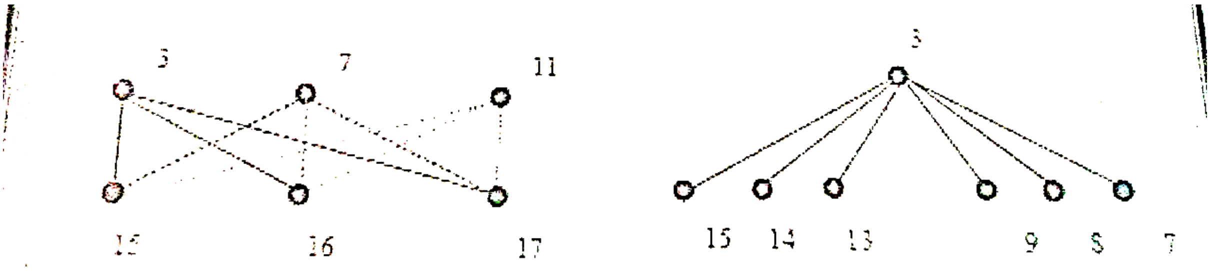


Figure 3.2: 3-sequential labelling for $K_{3,3}$ and $K_{1,6}$

The family of graphs known as windmill graphs m K_n ($n \geq 3$) consists of m copies of K_n that share a vertex. Let's refer to the network in the instance of n = 3 as a Dutch m-windmill since it consists of mK $_3$'s with a single vertex in common.

$m \equiv 0$ or 1 (mod 4) If and only if, the Dutch m-windmill is graceful.

we have mK$_4$'s with exactly one vertex in common., For n = 4. We call this kind of graphs as French m-windmill. The French m-windmill is graceful if $m \geq 4$.

Figure shows a graceful Dutch 5-windmill and a graceful French 4-windmill:

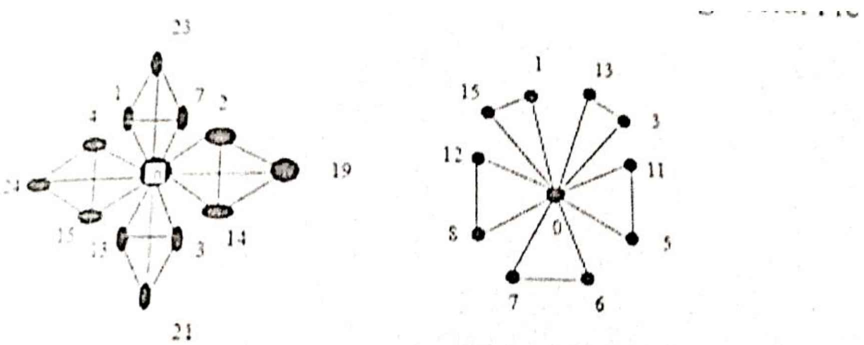


Figure 3.3 Graceful labeling of French 4-windmill and Dutch 5-windmill

3.1 General result

Statement 1 :There exists a partition of V called P = (A,B) where the number of edges with one end in A and the other in B is $\left[\frac{n}{2}\right]$ if G = (V,E) is elegant.

Proof : Assume that the graph G = (V, E) has a graceful labelling f. Then, take into account the partition P = (A, B) of V such that A = {u Є V: f(u) ≡ 0(mod 2) }. Since there are $\left[\frac{n}{2}\right]$ values in the range of 1 to n, and an odd difference can only be calculated by deducting an even value from an odd one, the number of edges joining two vertices with various parities must exactly $\left[\frac{n}{2}\right]$ equal the number of values in this range. Although Proposition 3 stipulates a prerequisite for the existence of a graceful labelling for a graph, it is useless in practise because it requires examining all 2m - 1 feasible partitions of V to determine whether a graph can support a graceful labelling.

Explanation 1:

If n > 0 and the degree of each vertex in a connected graph G is even, the graph is said to be Eulerian.

Theorem 1: Let G be an Eulerian graph. If n ≡ 1,2(mod 4) , then G is not graceful.

Proof: Assume that the graph G = (V, E) is a graceful Eulerian graph. Let C = (u_0,u_1... u_{n-1},u_n = u_0) be an Eulerian cycle of G and let f: V →[0, n] be a graceful labelling of G. If we add up C's edge labels modulo 2,we have

$$\sum_{i=1}^{n} f_{\gamma}(u_{i-1}u_i) = \sum_{i=1}^{n} | f(u_{i-1} - u_i)|$$

$$\equiv \sum_{i=1}^{n} f(u_{i-1}) - f(u_i) \equiv 0 \pmod 2$$

Since f is an graceful labelling of G and C is an Eulerian cycle, which means that the cycle C passes over each edge exactly once, we have:

$$\sum_{e \varepsilon E} f_{\gamma}(e) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \equiv 0 \pmod 2$$

Thus, to satisfy the above equation, we must have n $\equiv$ 0,3 (mod 4)

Unlike statement 1, the parity requirement offers a quick approach to determine whether or not an Eulerian graph can be graceful.

If n $\equiv$ 0 or 3 (mod 4) then G is a graceful Eulerian graph

Any graph with even degrees at each vertex is an Eulerian graph; such a graph need not be interconnected. For instance, K_5 and C_5 are Eulerian, but due to their respective 10 and 5 edge counts, they are not graceful.

In graph theory, it makes sense to consider substructures that prevent a graph from exhibiting a particular attribute, in this case grace. These substructures are known as forbidden substructures for the graph class and can be subgraphs, induced subgraphs, or other types. Therefore, one can consider looking for forbidden substructures for the category of graceful graphs.

Theorem 2. Every graph is an induced subgraph of a graceful graph.

Proof: Using a graph G = (V,E) as a starting point, create a graceful graph H from G such that G is an induced subgraph of H. Take into account a vertex labelling f: V→ [0, k] Injective for some $k \geq n$ such that u, v Є V exist with f(u) = 0 and f(v) = k. Additionally, the edge labelling f_γ : E → N is likewise injective. Let (x_1, x_2,... x_r) represent the collection of deleted edge labels. Without losing generality, x_1, x_2,..., x_s are not vertex labels, whereas x_{s+1},..., x_r . Add an edge linking w_i to u and a vertex labelled x_i for each x_i; $1 \leq i \leq s$ so that $f_\gamma(uw_i) = x_i$. Add a vertex w_i with the label $k + x_i$ for each x_i, , $s+1 \leq I \leq r$, then connect w i to u and v so that Both f_γ $(uw_i) = k + x_i$ and $f_\gamma(vw_i) = x_I$ are equal to k + x i. Be aware that by producing vertex labels with values greater than k in the previous phase, more missing edge labels may have been generated. These new edge labels that are lacking, nevertheless, are not vertex labels.

In order to ensure that $f_\gamma(uz_y) = y$, create a new vertex z_y with label y for each additional edge label that is absent. G is an induced subgraph in the graceful graph H that is the consequence.

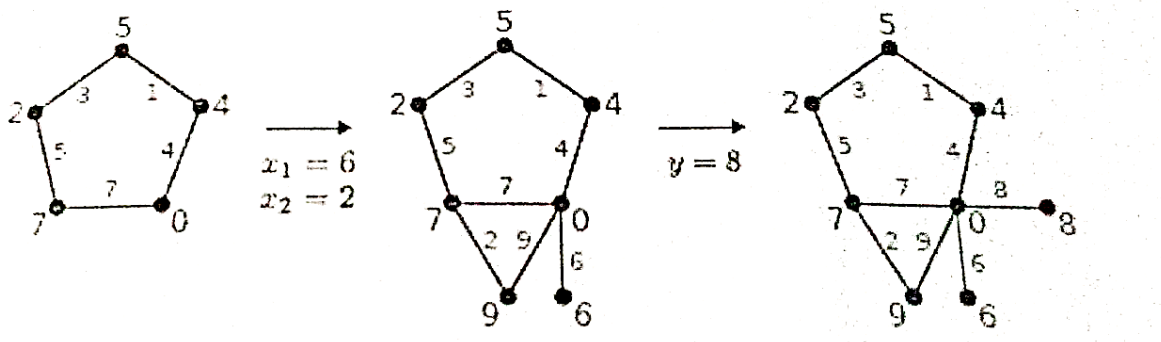


Figure 3.4: Constructing a graceful graph from C_5.

According to Theorem 2, the non-gracefulness of a graph G for graphs for which G is an induced subgraph is irrelevant. Additionally, it asserts that any graph may always be transformed into a graceful graph.

Two families of graphs the path graphs and the complete graphs have each had their gracefulness described thus far. For $m \geq 5$, the second is a family of non-graceful graphs, while the first is a family of graceful graphs. Additionally, we have demonstrated that any graph, graceful or not, can be used to create a graceful graph.

Explanation 2:

The full augmentation of a graceful graph G = (V, E) is the addition of an isolated vertex to G for each vertex label not used. Formally, $G_f = G \cup I_{m-n+1}$ is the full augmentation of G. Clearly, G_f also graceful and, in particular, graceful trees are already full augmented.

3.2 Gracefulness of graph classes

We demonstrate the gracefulness of certain graph classes in this section. The majority of findings supporting a graph class's gracefulness are provided by explicit graceful labelings. There are only a few tools for the lack of gracefulness in a graph class. In essence, we just have Theorem 1 and Proposition 3. We can also demonstrate by attempting to identify the graph and spotting a disagreement.

Statement 2: The cycle graph C_m is graceful if, and only if, $m \equiv 0,3 \pmod 4$.

Proof : Cycle graphs are Eulerian graphs. As a result, according to the parity condition, , C_m not graceful if $m \equiv 1,2 \pmod 4$. If not, let's use the $V(C_m) = \{u_0, u_1, \ldots, u_{m-1}\}$ such that $u_k u_{k+1} \in E(C_m)$ for $0 \leq k \leq m-1$ and $u_m = u_0$

If $m \equiv 0 \pmod 4$, then label the vertices according to the following formula:

$$f(u_i) = \begin{cases} \frac{i}{2} & if\ i = 0,2,4,\dots,m-2 \\ m - \frac{i-1}{2} & if\ i = 1,3,5,\dots,\frac{m}{2}-1 \\ m - \frac{i-1}{2} - 1 & if\ i = \frac{n}{2}+1\ ,\ \frac{n}{2}+3,\dots m-1 \end{cases}$$

If m ≡ 3 (mod 4), then label $V(C_m)$ as follows:

$$f(u_i) = \begin{cases} \frac{i}{2} & if\ i = 0,2,4,\dots,m-1 \\ n - \frac{i-1}{2} & if\ i = 1,3,5,\dots,\frac{m+1}{2}-1 \\ m - \frac{i-1}{2} - 1 & if\ i = \frac{m+1}{2}+1\ ,\ \frac{m+1}{2}+3,\dots m-2 \end{cases}$$

Remember that the cycle graphs' grace is characterised by the gracefulness of cycle graphs. A cycle graph Cp and a singleton graph K1 are joined to form the wheel graph W_p, which has the formula $W_p = C_p + K_1$

Statement 3. The wheel graph W_p graceful for all p ≥ 3.

Proof: Take into consideration the following two scenarios. Let $V(W_p) = \{u_0, u_1, \dots, u_{p-1}, v\}$ be the collection of vertices where v is the vertex linked with the cycle.

1. The following fomula provides a gracefull labelling If p ≡ 0 (mod 2):

$$f(v)=0$$

$$f(u_i) = \begin{cases} 2p & if \quad i = 0 \\ 2 & if \quad i = p - 1 \\ i & if \quad i = 1,3,5 \ldots, p - 3 \\ 2p - i - 1 & if \; i = 2,4,6 \ldots, p - 2 \end{cases}$$

2. If p ≡ 1 (mod 2), the labelling produced by the following formula is graceful:

f(v)=0

$$f(u_i) = \begin{cases} 2p & if \quad i = 0 \\ 2 & if \quad i = 1 \\ p + i & if \quad i = 2,4,6 \ldots, p - 1 \\ p + 1 - i & if \quad i = 3,7, \ldots, p - 2 \end{cases}$$

A tree that becomes a caterpillar after all of its leaves are taken off is one that has a path graph.

Statement 4: All caterpillar trees are graceful.

Proof : Create a planar bipartite representation of the caterpillar tree and label it as in Figure 2.4. Verifying that such a drawing approach is always possible. n here stands for a vertex

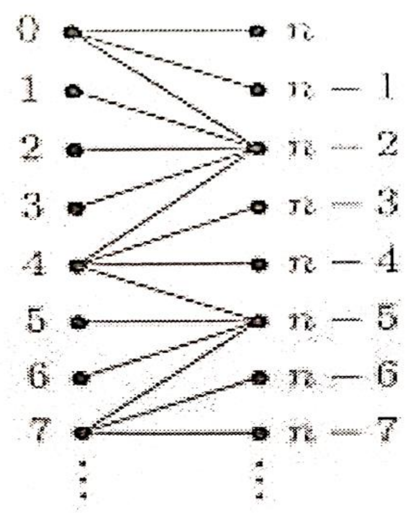


Figure 3.5: Graceful labeling of caterpillar tree

Note that a path graph P_n is also a caterpillar tree and the labelling scheme given by Proposition 5 , when applied to a path graph, yields the same labelling constructed before.

The complete bipartite graph $K_{p,q}$ is a bipartite graph G = (A, B, E) such that |A|= p, |B| = q , and if u Є A and v Є B , then uv Є E. In particular, the star graph is the complete bipartite graph $K_{1,q}$.

Statement 5: The complete bipartite graph $K_{p,q}$ is graceful for all p,q ≥ 1

Proof: Let G = (A, B, E) be a bipartite graph with a = |A| and b = |B|. Assign the vertices from A with numbers 0, 1 ,...,a-1 and assign the vertices from B with numbers a, 2a,...,ba.

CHAPTER-4

APPLICATIONS OF GRACEFUL GRAPHS

Labelled graphs serve as useful tools for a broad range of applications. One of the most used approaches for labelling graphs is the odd graceful labelling [18]. The labelling of graphs is thought to be primarily a theoretical topic in the fields of discrete mathematics and graph theory, but it really has a wide range of applications, some of which are described below.

The coding theory:

According to the theory of coding, creating some significant classes of effective non- periodic codes for pulse radar and missile guidance is similar to labelling the entire graph in a way that makes each edge stand out. The time positions at which pulses are transmitted are then determined by the node labels.

The x-ray crystallography:

When an X-ray beam impacts a crystal, it diffracts into a variety of distinct directions making X-ray diffraction one of the most effective methods for identifying the structural characteristics of crystalline materials. Sometimes, the same diffraction data is present in many structures. This issue is mathematically identical to figuring out all of the labelling for the relevant graphs that result in a predetermined set of edge labels.

Addressing the communication network:

A communication network is made up of nodes, each of which has computer power and can send and receive messages across wired or wireless communication links.

A few examples of the fundamental network topologies are completely linked, mesh, star,ring, tree, and bus. A single network may be made up of a number of connected subnets with various topologies. In this work, these difficulties are briefly examined.

Local Area Networks (LAN), which include networks within a single b building, and Wide Area Networks (WAN), which include networks between buildings, are further categories for networks. Each user terminal could be given a "node label," provided that all connecting "edges"(communication links) have unique labels. Inthis method, the numbers of any two communicating terminals immediately identify the link label of the connecting path (by simple subtraction), and vice versa, the label of the connecting path specifically specifies the pair of user terminals that it connects.

In this section, we restrict our discussion to application of graceful labeling in dental arc.

GRACEFUL LABELING OF A DENTAL ARCH

When an edge xy is assigned to a label l f(x)f(y) l and the resultant edge labels are distinct, then function f is an injection from the vertices of graph G to the set {0,1.,q}

such labelling's are termed graceful.

The mandibular or lower arch and the maxillary, or upper arch, are two different parts of the dental arch. First premolars, second premolars, canines, molars, and right and left central incisors make up each arch. Here, it is taken into account up until the first molars. On each side of the dental arch,there are six teeth overall, for a total of 12 teeth. Every tooth in the arch is regarded as a vertex, and a line connecting the teeth on the left and right sides of the same kind, as well as the adjacent teeth, forms the borders of the arch. In this graph, graceful labelling is used using the vertex set and edge set of V=0,1,.......16and E=1,2,.......16, respectively. Vertex labels and edge labels are discovered to be separate during labelling. The layout of the vertex labels also follows a pattern. Graph labelling is used to evaluate how the various teeth relate to the arch.

K-GRACEFUL LABELING

A graph G with q edges is k-graceful if it has labelling f from its vertices of G to{0,1,2....q+k-1} such that the set of edge labels produced by the absolute value of the label difference between adjacent vertices is k,k+1,.......q+k-1.

V= {0,1,2,....q+k-1}

E = {k, k+1,....q+k-1}

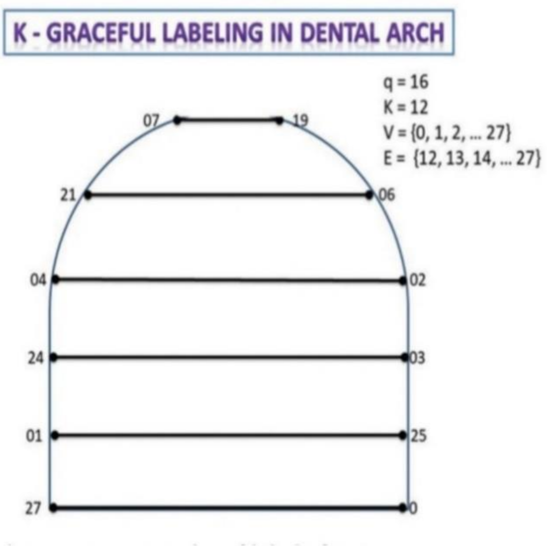
where q = edges and k = vertices.

12 – GRACEFUL LABELING

q = 16

k = 12

The vertex set has labels ranging from 0 to q+k-1 = (16+12)-1 = 27



The edge set is made up of k,k+1,....q+k-1 i.e., labels beginning with 12 and ending with 27. Therefore,

V= {0,1,....27} and

E= {12,13,14,....27}

Therefore, k - Graceful Labelling satisfying its requirements can be used to represent the dental arch.

ODD GRACEFUL LABELING

If there is an injection f from V(G) to { 0, 1, 2,.... 2q-1} such that when each edge x and y are assigned label l f(x) and f(y) l the resulting edge labels are { 1, 3, 5,.... 2q-1}

then the graph G with q edges is said to be odd- graceful. Now, we're attempting to devise a graceful labelling system for the dental arch. While labelling of graphs is thought to be a major theoretical issue in the field of graph theory and it serves as models in a wide range of applications, odd graceful labelling is one of the most popular labelling methods of the graph.

Here,

p=12 q=16

The vertex set has labels ranging from

0 to 2q – 1= (2*16)-1 =31

The edge set consists of all the odd labels <31

Therefore, V= {0,1,....31} and E= {1,3,5,....31}

In dental arch models, odd elegant labelling is therefore relevant.

INFERENCE

Examining the use of graph labelling in the dental industry is the main goal of this article. Graceful labelling may be used to depict the dental arch, and we discover that this process follows a specific pattern. To examine the arch's teeth, one could utilise this pattern. As a result, graceful labelling is an effective tool that enables the quick and convenient learning of complex patterns across a variety of disciplines.

CONCLUSION

Graph theory is a member of discrete mathematics. Graph Theory has a number of applications in many field. Here in this project, we give a brief study about the graceful graphs and one of its applications. In the opening section we give ideas regarding graph and its preliminaries. In second chapter we define graceful graph some of its basic definition and also about different type of labelling. And in third chapter we discuss about complete graceful graph. Later we give some general result regarding the graceful graph. In the last chapter we discuss about the application of graceful labelling in Dental Arch.

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