INTERPOLATION

. DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR

Bachelor's degree in Mathematics

Jointly by,

ANGEL ROY	REG.NO:200021033164
ARJUN SURESH	REG. NO:200021033166
FARHIN V A.	REG. NO:200021033169
JISMY NAZAR	REG. NO:200021033170
SHADHIL SALAM	REG. NO:200021033180

Under the supervision of

. MISS.ALAKA MOHAN

Department of Mathematics

Bharata Mata College

Thrikkakara, 2020-2023

Affiliated to Mahatma Gandhi University,

Kottayam

ACKNOWLEDGEMENT

We like to express our sincere thanks to Miss. ALAKA MOHAN ,Associate Professor, Department of Mathematics , Bharata Mata College , Thrikkakara whose guidance helped us to complete this project work . We are also thankful to Dr. SEETHU VARGHESE, Head of The Department Of Mathematics for her helping attitude and valuable advices.

We are obligated to all our teachers of Department of Mathematics for their encouragement and co-operation for completing this work.

Place : Thrikkakara

Date : -04-2023

Angel Roy

Arjun Suresh

Farhin V A

Jismy Nazar

Shadhil Salam

CERTIFICATE

This is to certify that the Dissertation entitled "**INTERPOLATION**" submitted jointly by Miss. Angel Roy, Miss. Farhin.V A, Miss. Jismy Nazar, Mr. Shadhil Salam, Mr. Arjun Suresh in partial fulfilment of the requirements for the BSc. Degree in Mathematics is a bonafide record of the studies undertaken by them under my supervision of the Department of Mathematics, Bharata Mata College, Thrikkakara during 2020-2023. This dissertation has not been submitted for any other degree elsewhere .

Place : Thrikkakara

Date : 04-2023

Miss. ALAKA MOHAN

Supervisor

DECLARATION

We hereby declare that this project entitled "**INTERPOLATION**" is a bonafide record of work done by us under the supervision of Miss. ALAKA MOHAN and the work has not previously formed the basis for any other qualification, fellowship , or other similar title of other university or board.

Place : Thrikkakara

Date : - 04 – 2023

Angel Roy

Farhin V A

Jismy Nazar

Shadhil Salam

Arjun Suresh

CONTENTS

ABSTRACT

Chapter 1 includes the preliminaries about interpolation such as forward difference , backward difference , central difference and errors in interpolation.

Chapter 2 discuss about different methods of interpolation and some other formulas .

Chapter 3 includes interpolation with unequal intervals, divided differences ,lagrange ‘s interpolation for unequal intervals .

Chapter 4 discuss about spline interpolation and error .

INTRODUCTION

Interpolation is the process of estimating the value of a function for any intermediate value of the independent variable, given a set of tabulated values of the function at certain values of the independent variable. If the function is known explicitly, then the value of the function can be easily found for any value of the independent variable. However, if the form of the function is not known, then interpolation is used to estimate the value of the function at any intermediate value of the independent variable.

Polynomial interpolation is a common method used for interpolation, where a polynomial function is used to approximate the unknown function. The process involves finding a polynomial function that passes through the given set of tabulated values of the function at certain values of the independent variable. The polynomial function is then used to interpolate or extrapolate the value of the function at any intermediate or outside values of the independent variable.

The calculus of finite differences is a useful tool in the study of interpolation. By taking forward or backward differences of a function, we can derive interpolation formulae that are commonly used in engineering and scientific investigations. These formulae can be used to approximate the value of the function at any intermediate value of the independent variable.

Overall, interpolation is an important technique used in many areas of science and engineering. Polynomial interpolation and the calculus of finite differences are important tools in the study of interpolation and are commonly used in practical applications.

1.PRILIMINARIES

1.1 TRANSCENDENTAL EQUATTIONS

If h(u) is a Polynomial that is

$h(u) = a_0 u_m + a_1 u_{m-1} + a_2 u_{m-2} + \dots + a_{m-1} u + a_m$

Then the equation is called an algebraic equation

A non - algebraic equations are called transcendental equations. That is ,equations involving functions like, sin u ,cos u tan u, log u, e^u , etc… are called transcendental equations.
Example: $2e^u+1=0$

1.2 INTERPOLATION

Given a set of data point (u_j, h(u_j)) for 0 < j < m where the nature of the function h(u) is not known explicitly it is required to find a simple function Ψ(u) such that, h(u) and Ψ(u) agree at the set of data points, such process is called interpolation .

If Ψ(u) is a Polynomial, then the process is called polynomial interpolation and Ψ(u) is called the interpolating polynomial.

1.2.1 ERRORS IN POLYNOMIAL INTERPOLATION:

Let v(u) be a function defined by the (m+1) points (u_j,v_j),j=0,1,2,…m and v(u) be a continuous function and differentiable (m+1) times.

Let v(u) be approximated by a Polynomial $\Psi_m(u)$ of degree not exceeding m such that $\Psi_m(u_j)=v_j$, j=0,1,2….,m. we know that $\Psi_m(u_j) - v(u_j) = 0$,j=0,1,2,…,m $v(u)-\Psi_m(u)=\pi$______$_{m+1}(u)\ v_{m+1}$(£) $(n-1)!$

where, $u_0 <£< u_m$

1.3 FINITE DIFFERENCES

Assume that we have a table of values (u_j, v_j), j=0,1,2,…m of any function $v = h(u)$, the values of u being equally spaced in. that is $u_j = 0,1,2,3, \ldots . m$

Suppose that we are required to recover the values of h(u) for some intermediate values of u, or to obtain the derivative of h(u) for some u,$u_0 < u < u_m$. The methods for the solution to these problems are based on the concept of the differences of a function there are 3 types of difference of u.

•Forward Differences

•Backward Differences

•Central Differences.

1.3.1 FORWARD DIFFERENCES:

If v_0, v_1,v_2,… . y_n Denote s set of values of v. The differences $v_1 - v_0$, $v_2 - v_1$, … … . , $v_m - v_{m-1}$

Are called first forward differences. If they are denoted by Δv_0, Δv_1, … … , Δv_{m-1}

Respectively ,so that if $\Delta v_0 = v_1 - v_0$,$\Delta v_1 = v_2 - v_1$,… … … . . , $\Delta v_{m-1} = v_m - v_{m-1}$

Where , Δ is called the forward are called second forward differences and are denoted by$\Delta^2 v_0$,$\Delta^2 v_1$, ….

We can define third forward differences ,fourth etc.

1.3.2 FORWARD DIFFERENCE TABLE:

u	v	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
u_0	v_0						
		Δv_0					
u_1	v_1		$\Delta 2v0$				
		Δv_1		$\Delta 3v0$			
u_2	v_2		$\Delta 2v1$		$\Delta 4v0$		
		Δv_2		$\Delta 3v1$		$\Delta 5v0$	
u_3	v_3		$\Delta 2v2$		$\Delta 4v1$		$\Delta^6 v_0$
		Δv_3		$\Delta 3v2$		$\Delta 5v1$	
u_4	v_4		$\Delta 2v3$		$\Delta 4v2$		
		Δv_4		$\Delta 3v3$			
u_5	v_5		$\Delta 2v4$				
		Δv_5					
u_6	v_6						

1.3.3 BACKWARD DIFFERENCES

The differences $v_1 - v_0$, $v_2 - v_1$,... . , $v_m - v_{m-1}$ are called first backward differences if they are denoted by ∇v_1,∇v_2,... ∇v_m respectively, so that $\nabla v_1 = v_1 - v_0$,$\nabla v_2 = v_2 - v_1$,... $\nabla v_m = v_m - v_{m-1}$ Where ∇ is called the backward difference operator .

1.3.4 BACKWARD DIFFERENCE TABLE

u	v	∇	∇^2	∇^3	∇^4	∇^5	∇^6
u_0	v_0						
		∇u_1					
u_1	v_1		$\nabla 2v2$				
		∇v_2		$\nabla 3v3$			
u_2	v_2		$\nabla 2v3$		$\nabla 4v4$		
		$\nabla v3$		$\nabla 3v4$		$\nabla 5v5$	
u_3	v_3		$\nabla 2v4$		$\nabla 4v5$		$\nabla 6v6$
		∇v_4		$\nabla 3v5$		$\nabla 5v6$	
u_4	v_4		$\nabla 2v5$		$\nabla 4v6$		
		∇v_5		$\nabla 3v6$			
u_5	v_5		$\nabla 2v6$				
		∇v_6					
u_6	v_6						

1.3.5 CENTRAL DIFFERENCES

The central difference operator δ is defined by the relations

$$v_1 - v_0 = \delta v\tfrac{1}{2}, v_2 - v_1 = \delta\, v\tfrac{3}{2}, \ldots, \quad v_m - v_{m-1} = \delta\, v_{m-12}$$

Similarly higher order central differences can be defined.

1.3.6 CENTRAL DIFFERENCE TABLE

u	v	δ	δ^2	δ^3	δ^4	δ^5	δ^6
u_0	v_0						
		$\delta v\ 1/\bar{2}$					
u_1	v_1		$\delta^2\ v_1$				
		$\delta v\ 3/\bar{2}$		$\delta^3 v\ {}_{3/\bar{2}}$			
u_2	v_2		$\delta^2\ v_2$		$\delta^4 v_2$		
		$\delta v\, 5/\bar{2}$		$\delta^3 v\ {}_{5/\bar{2}}$		$\delta^5\ v\ {}_{5/\bar{2}}$	
u_3	v_3		$\delta^2\ v_3$		$\delta^4 v_3$		$\delta^6 v_3$
		$\delta v\, 7/\bar{2}$		$\delta^3 v\ {}_{7/\bar{2}}$		$\delta^5 v\ {}_{7/2}$	
u_4	v_4		$\delta^2\ v_4$		$\delta^4 v\ 4$		
		$\delta v\, 9/\bar{2}$		$\delta^3 v\ {}_{9/2}$			
u_5	v_5		$\delta 2 v 5$				
		$\delta v\ {}_{11/2}$					
u_6	v_6						

1.4 SYMBOLIC RELATIONS AND SEPARATIONS OF SYMBOLS

1.4.1 AVERAGE OPERATION

The averaging operator ‘μ' is defined by the equation $\mu v_t = {}_{2-}(v_{t+12} + v_{t-12})$

1.4.2 SHIFT OPERATION

The shift operator E is defined by the equation E$v_t = v_{t+1}$

1.4.3 FORMULAS

- $\Delta = E - 1$
- $\nabla = 1 - E^{-1}$
- $\delta = E_{1/2} - E_{-1/2}$
- $\mu = \frac{1}{2} (E_{1/2} - E_{-1/2})$
- $\mu 2 = 1 + \frac{1}{4}\delta^2$
- $E = e^{dD}$

1.5 NEWTON'S FORMULA FOR INTERPOLATION

1.5.1 NEWTON'S FORWARD DIFFERECNCE INTERPOLATION FORMULA

Given a set of (m+1) values namely (u_0, v_0), (u_1, v_1), ... (u_m, v_m) Of u and v.
We have to find a Polynomial of the m^th degree v_m (u).that is v and v_m(u) agree the tabulated points let the values of u be equidistant .that is, Let, $u = u_0 + kd$.
Therefore,

$$v_m(u) = v_0 + k\Delta u_0 + k(k-1)/1!\Delta^2 v_0 + k(k-1)(k-2)/2!\,\Delta^3 v_0 + \cdots + k(k-1)(k-2)\ldots(k-m+1)/2!\Delta^m v_0$$

Which is Newton's forward difference interpolation formula. this formula is useful for interpolation near the beginning of a set of tabular values .

1.5.2 ERROR IN NEWTON'S FORWARD DIFFERENCES INTERPOLATION FORM

$$v(u) - v_m(u) = k(k-1)((k-2)\ldots!.(k-m)/\text{m}!\,\Delta^{m+1} u(\alpha) \quad , u_0 < \alpha < u_m$$

1.5.3 NEWTON'S BAKCWARD DIFFERENCE INTERPOLATION FORMULA

Given a set of (m+1) values , ,(u_0, v_0), (u_1, v_1)... (u_n,v_n) Of u and v. We have to find a Polynomial of the m^th degree $v_m(u)$such that v and v_m agree at the tabulated points u_m, u_{m-1}, u_{m-2} ... u_1, u_0 Let $u = u_n + kd$

Therefore ,

$$v_m(u) = v_m + k\nabla v_m + k(k+1)/2!^* \nabla^2 v_m + \ldots\ldots\ldots\ldots + \quad k(k+1)\ldots.(k+m-1)\,\nabla_m v_m$$

1.5.4 ERROR IN NEWTON'S BACKWARD DIFFERENCE INTERPOLATION FOR

$v^{(u)} - v_m(u) = \frac{k(k+1)(k+2)...(k+m)}{(m+1)!}\nabla_{m+1} v(\alpha)$

2. METHODS OF INERPOLATION

2.1 LAGRANGE'S INTERPOLATION AT REGULAR INTERVAL

Let " d " be considered as regular intervals . Then;

$b_{i+1} - b_i = d$ $\quad i=1,2,\ldots\ldots,m-1$ ---- (1)

for easy computation it is common to take 'm' odd let ,

$u = b_t + d_n$ ---- (2)

Here t =(m+1)/2. Thus n=0 equal to the center of the interval of tabular points. From the equation (2) $P_n(u)$ and $L_i(u)$ be taken as function of n . From this

$$L_i(u) = \frac{(u-b_1)\ldots(u-b_{i-1})\,(u-b_{i+1})\ldots(u-b_m)}{(bi-b_1)\ldots(b_i-b_{i-1})(b_i-b_{i-1})\ldots(b_i-b_m)} \quad \text{-------(3)} \quad \text{[L.I.F]}$$

Which denote $L_i(m)$ is independent of d and thus can be tabulated as function of 'n'. When we use equation (3) and write $h(b_t + d_n)$ as h(n) the formula of Lagrange's formula become

$$h(n)= \sum^{m}_{i=1} L_i(n)\, h(b_i) + d^m k_m(n)/m!\ \ h^{(n)}{}_{(\varsigma)} \quad \text{----- (4)}$$

$$P_m(n) =(n-t +1)\,(n-t +2)\ldots n(n+1)\ldots(n+t-1) \quad \text{-----(5)}$$

2.1.1 FINITE DIFFERENCE

Finite differences are used extensively in numerical solutions of partial defferential equation and boundary valued problems of ordinary D.E. Using the regular interval points we can define ;

a) ***The P^{th} forward difference of h(u)***

$\Delta^P h(u) = \Delta^{P-1}h(u+d) - \Delta^{P-1}h(u)$ $\quad$ P=1,2,3…… $\quad$ $\Delta^o h(u) = h(u)$

------(6)

Example,

$\Delta^1 h(u) = \Delta h(u)= h(u+d) - h(u)$ -----(7)

$\Delta^2 h(u)= \Delta h(u+d)- \Delta h(u)=h(u+2d)-2h(u+d)+h(u)$ ----(8)

b) ***The P^{th} backward difference***

$\nabla^P h(u) = \nabla^{P-1} h(u) - \nabla^{P-1}h(u-d)$ $\quad$ P = 1, 2…… ------(9

$\nabla^{O}h(u) = h(u)$

C) The P^{th} Central Difference

$ƒ^{P}h(u) = ƒ^{P-1} h(u + ½ d)$ P = 1,2…..

$ƒ^{O} h(u) = h(u)$

From Newton's forward formula $V(n) = h_o + (n)_1 \Delta h_o + (n)_2 \Delta^2 h_o + \ldots$

$+(n)_m \Delta^m h_o = \sum^{m}_{i=0} (n)_i \Delta^i h_o$

We already knew that this is algebraically equivalent to lagrangian's interpolation formula at regular interval for m+1 point .$b_{o,\ldots\ldots}b_n$ $(n)_m$ is a polynomial of difference degree m in n. from equation (7) and (6) we equation of finite difference interpolated formula as ;

$V(j)b=\sum_{i=0m} (j)_I \Delta_i h_o = \sum_{mi=0} \sum_{mp=0} (-1)_{i-P} (I \ \ P) h_k$

$\sum_{m\ p=o} \sum_{mi=p} (-1)_{\ i-P} (j,i) (I, P) h_k$ -------------(8) j=0….n

2.2 ITERATED INTERPOLATION

Iterated interpolation is a sequence of interpolant which is used to overcome the disadvantage of Lagrange's formula and which is an advantage of finite – difference interpolation formula .By using this formula ,in which a

sequence of interpolants in the Lagrange's context when going from m to m+1 point. By $V_{m1},\ldots,m_k(u)$ the Lagrangian's interpolation formula using the point $b_{m1},\ldots\ldots,b_{mp}$. It do not want to be equally spaced .

Then it can be write it as ;

$V_{1,2,\ldots m}(u) = 1/b_m-b_{m-1}$ $| V_{1,2,\ldots\ldots,m}(u)\ b_{m-1}-u|$--------------(1)

$|V_{1,2,\ldots.m}(u) \quad b_m-u\ |$

This can be corrected by looking R.H.S. which is polynomial of degree m-1 then take the value $h(b_j)$ at point $b_{j\ \text{where } j=1,\ldots,n}$.

Equation (1)indicates how lagrangian's formula of order m can be generated from lower order formula that given below will give the general formula for equation(1). $V_j(u) = h(b_j)$

b_1 b_1-u $v_1(u)$

b_2 b_2-u $v_2(u)$ $v_{=1,2}(u)$

b_3 b_3-u $v_2(u)$ $v_{=1,3}(u)$ $v_{1,2,3}(u)$

…………………………………………………………………………

b_n b_m-u $v_m(u)$ $v_{1,m}(u)$ $v_{1,2,3}(u)$ ………. $v_{1,2,\ldots\ldots m}(u)$

From this we come to know that we can generate each entries of the table by the previous column by analogy with equation (1)

2.3 INVERSE INTERPOLATION

From previous chapters we shall know about the solution for general equation h(u) = 0 ,the basics of the solution be inverse interpolation . the solution h(u)=0 is an example for numerical problem of finding the zeroes .In inverse interpolation we see the powerful and straight forward way to find out such zeroes of function .

The functions whose zero we going to find be v = h(u) and suppose it is tabulated at a series of point then we have;

u	u1	u2	………	un
v=h(u)	$h(u_1)$	$h(u_2)$	……….	$h(u_n)$

Suppose an interval $[u_1, u_2]$,h(u) satisfies condition of inverse function theorem . In other way that h'(u) ≠ 0 then we can write u= s(v) where s is the inverse function to h .

Therefore , finding values of s(0) is equal to finding a zero of h(u). to calculate s(0) we change the above table as

v	$h(u_1)$	$h(u_2)$	………	$h(u_n)$
u= s(v)	u_1	u_2	…….	u_n

In interpolation let $h(u_1)$ …… $h(u_n)$ be the tabular point of independent variable v [in regular interval] then let u_1,……..,u_n be the function valued point .

When we use lagrangian's interpolation formula to approximate s(v) by a polynomial and then interpolate at point v=0 we get approximate to β = s(0).

2.4 HERMITE INTERPOLATION

In this section we consider n_i= 1 i= 1,…….,t we take the 1^{st} derivative as well as the function is known at t of m tabular point

$h(u) = \sum^{m}_{i=1} f_i(u)\, h(b_i) + f_i(u)\, h'(b_i) + M(u) = v(u)+M(u)$---------(1) $v(u) = \sum\ f_i(u)\, h(b_i) + \sum$

$f_i(u)\, h'(b_i)$ ---------(2)

$f_i(u)\ f_i^{-}(u)$ are both polynomials we required the error term $M(u)$
Such that, $M(b_i) = 0$ $i = 1,2,\ldots m$ $M'(b_i)$
$= 0$ $i = 1,2,\ldots s$

the following condition satisfies the $f_i(u)$ and $f^{-}(u)$

$f_i(b_t) = f_{it}$ $i,t = 1,\ldots,m$

$f^{-}_i(b_t) = 0$ $i = 1\ldots,s$; $t = 1,\ldots,m$ $f_i(b_t) = 0$ i

$= 1,\ldots,n$; $t = 1,\ldots,s$ $f^{-}(b_t) = f_{it}$ $i,t = 1,\ldots,s$

The interpolation formula become ;

$h(u) = \sum f_i(u)\, h(b_i) + \sum f^{-}_i(u)\, h'(b_i) + p_m(u)\, p_t(u)/(m+t)!\ \ h^{(m+t)}(\S)$ ---------(3) $\{1-(u-b_i)$
$[l'_{it}(b_i)]\}i_{im}(u)l_{it}(u)$

$f_i(b_t) =$ $i= 1,\ldots,t$ $l_{it}(u)\ p_t(u)/$

$p_t(b_i)$ $i=t+1,\ldots, m$

Modified interpolation formula ;

$F_i^{-}(u) = (u-b_i)l_{it}(u)\ \ l_{im}(u)$ $i = 1,\ldots,t$

When t=m The formula is ;

$$h(u) = \sum f_i(u)\, h(b_i) + \sum f^{-}_i(u)\, h'(b_i) + p_m^2(u)/(2m)!\ \ * h^{(2m)}(\S) \text{ --------(4)}$$

Equation (4) is called hermite interpolation formula and it is also known as oscIlatory interpolation .

2.5 CENTRAL DIFFERENCE FOR INTERPOLATION FORMULA

We discussed about Newton's forward and backward interpolation formulae which is applicable at the starting and end of tabular values . the central difference formulae is the most suited for interpolation for tabular value set .

Most important central difference formulae are given below

2.5.1 GAUSSES CENTRAL DIFFERENCE FORMULA

(a) ; *Gauss's forward formula* : In the below table the central value is tooked as v_o corresponding to $u = u_o$ .The difference used in this formulae are aligned as a line in the table given below . the formula is ;

$V_p = v_o + Q_1\, \Delta v_o + Q_2\, \Delta^2 v\text{-}1 + Q_3\, \Delta^3 v\text{-}1 + Q_4\, \Delta v^4\text{-}1 + \ldots\ldots$

Where $Q_1, Q_2, \ldots\ldots$ has to be determined. The v_p of left can be denoted in terms of v_o, Δv_o and higher order differences of v_o as given below

u	v	Δ	Δ2	Δ3	Δ4	Δ5	Δ6
u-3	v-3						
u-2	v-2	Δv-3	Δ2 v-3	Δ3 v-3			
u-1	v-1	Δv-2	Δ2 v-2	Δ3 v-2	Δ4 v-3	Δ5 v-3	Δ6 v-3
u o	v o	Δv-1	Δ2 v-1	Δ3 v-1	Δ4 v-2	Δ5 v-2	
u 1	v 1	Δvo	Δ2 v0	Δ3 v0	Δ4 v-1		
u 2	v 2	Δv1	Δ2 v1				
u 3	v 3	Δv2					

$v_p = £^p v_o$

$= (1+\Delta)^2 v_o$

$= v_o + p\Delta v_o + p(p-1)/2! \ * \Delta^2 v_o + p(p-1)(p-2)/3! * \Delta^3 v_o + \ldots.$

Then the right side become v_o, Δv_o and higher order differences .we have ; $\Delta 2v_{-1} = \Delta 2 £^{-1} v_o$

$= \Delta^2 (1+\Delta)^{-1} v_o$

$= \Delta^2 (1-\Delta + \Delta^2 - \Delta^3 + \ldots..) v_{o\Delta}$

$= \Delta^2 v_o - \Delta^3 v_o + \Delta^4 v_o - \Delta^5 v_o + \ldots..$

$\Delta 3v_{-1} = \Delta 3vo - \Delta 4vo + \Delta 5\ vo - \Delta 6\ vo + \ldots\ldots$

$\Delta 4v_{-2} = \Delta 4vo - 2\Delta 5\ v0 + 3\ \Delta 6\ vo + \ldots$ Since the equation becomes $v_o + p\Delta v_o + p(p-1)/2! \ * \Delta^2 v_o + p(p-1)(p-2)/3! * \Delta^3 v_o + p(p-1)(p-2(p-3)/4! * \Delta^4 v_o + \ldots.$

$= v_o + Q_1 \Delta v_o + Q_2 (\Delta^2 v_o - \Delta^3 v_o + \Delta^4 v_o - \Delta^5 v_o + \ldots\ldots..) + Q_3 (= \Delta^3 v_o - \Delta^4 v_o + \Delta^5 v_o - \Delta^6 v_o + \ldots.) + Q_4 (\Delta^4 v_o - 2\Delta^5 v_0 + 3\ \Delta^6 v_o - 4\Delta^7 v_o + \ldots.) + \ldots..$ ----------(1)

Equality of coefficient Δv_o, $\Delta^2 v_o$, $\Delta^3 v_o$ etc. in equation (1) we get

$Q_1 = p$ $\qquad$ $Q_2 = p(p-1)/2$

$- Q_2 + Q_3 = p(p-1)(p\Delta^2 v_o - 2)/3!$ Will give $Q_3 = (p + 1)\ p(p-1)/3!$ -----------(2) $Q_4 = (p + 1)\ p(p-1)\ (p-2)/4!$ etc.

b) ; Gauss' s Backward Formulae : It is used in the given table below by showing the differences

.   .

.   .

.   .

u_{-1} v_{-1} Δv_{-1} $\Delta^3 v_{-2}$ $\Delta^5 v_{-3}$

u_o v_o Δv_o $\Delta_2 v_{-1}$ $\Delta_3 v_{-1}$ $\Delta_4 v_{-2}$ $\Delta_5 v_{-2}$ $\Delta_6 v_{-3}$.

.

.   .

.   .

then Gauss's backward formula can be written as

$vp = vo + Q'1\Delta v\text{-}1 + Q2'\Delta 2v\text{-}1+Q3'\Delta 3v\text{-}2+Q4'\Delta 4v\text{-}2+\ldots..$

we have to findout Q'_1, Q_2',…here. As in the guass's forward formula we obtain

$Q'1 = p$

$Q_2' = p(p-1)/2!$ -----------(3)

$Q_3' = p(p-1)(p-2)/3!$

$Q_4' = p(p-1)(p-2(p-3)/4!$

2.5.2 STIRLING FORMULAE

Lets take the average of the gauss's backward and forward fmulae ,we get ;

$$v_p = v_o + p * (\Delta v_{-1} +\Delta v_o)/2 + p^2 /2 * \Delta^2 v_{-1} + p(p^2-1)/3! * \Delta^3 v_{-1} + \Delta^3 v_{-2}/ 2$$

$$+ p^2(p^2-1)/4! * \Delta^4 v_{-2} + \ldots.. \quad \text{-----------(1)}$$

This is termed as stirlings formulae

2.5.3 BESSEL'S FORMULA

For practicals interpolation this is commonly used . It use the differences showed in below table .In the bracket the mean value has token .

. .

. .

. .

u_{-1} v_{-1} $\Delta^2 v_{-1}$ $\Delta^4 v_{-2}$ $\Delta^6 v_{-3}$ u_o v_o

Δv_o $\Delta^3 v_{-1}$ $\Delta^5 v_{-2}$ u_1 v_1 $\Delta^2 v_{-1}$

$\Delta^4 v_{-1}$ $\Delta^6 v_{-3}$

. .

. .

It can assumed in the form ; $V_p = V_o + V_1 /2 + B_1 \Delta v_o + B_2 * \Delta^2 v_{-1} + \Delta^2 v_o / 2 + B_3 * \Delta^3 v_{-1} + B_4 * \Delta^4 v_{-2} + \Delta^4 v_{-1} /2$

$= V_o + [B_1 + ½] + B_2 * \Delta^2 v_{-1} + \Delta^2 v_o / 2 + B_3 * \Delta^3 v_{-1} + B_4 * \Delta^4 v_{-2} + \Delta^4 v_{-1} /2$

By the reference of gausses forward formulae

$B_1 + ½ = p$ $B_2 = p(p-1)/2!$

-----------(2)

$B_3 = p(p-1)[p – ½] /3!$ $B_4 = p(p-1)(p+1(p-1)/4!$

Then the formulae become

$V_p = V_o + p\Delta v_o + p(p-1)/2! * \Delta^2 v_{-1} + \Delta^2 v_o / 2 + p(p-1)[p – ½] /3! *\Delta^3 v_{-1}$

$+ p(p-1)(p+1(p-1)/4! * \Delta^4 v_{-2} + \Delta^4 v_{-1} /2 +.....$

2.5.4 EVERETT FORMULA

Here the even ordered differences are only used and interpolation formulae are given below ;

uo vo - Δ2 v-1 - Δ4 v-2 - Δ6 v-3 u1 v1

- Δ2 v-1 - Δ4 v-1 - Δ6 v-2 the formula is ;

$£_o = 1 – p = q$ $F_o = p$

$£_2 = q*(q^2 – 1^2) /3!$ $F_2 = p(p^2-1^2) /3!$

$£_4 = q*(q^2 – 1^2)(q^2 – 2^2)/5!$ $F_4 = = p*(p^2 – 1^2)(p^2 – 2^2)/5!$

…………

Hence the formulae become

$V_p = qv_o + q*(q^2 – 1^2) /3! *\Delta^2 v_{-1} + q*(q^2 – 1^2)(q^2 – 2^2)/5! * \Delta^4 v_{-2} + ...$

$+ pv_1 + p(p^2-1^2) /3! *\Delta^2 v_o + p*(p^2 – 1^2)(p^2 – 2^2)/5! *\Delta^4 v_{-1} +.....$ ----(1)

2.5.5 Relation between Basseles and Everetts formulae

$V_p = v_o + p\Delta v_o + p(p-1)/2! . \Delta^2 v_{-1} +\Delta^2 v_o /2 + p(p-1)[p – ½] /3! *\Delta^3 v_{-1}$

$+ p(p-1)(p+1)(p-2)/4! *\Delta^4 v_{-2} + \Delta^4 v_{-1} /2 +.....$

$= v_o + p(v_1-v_o) + p\Delta v_o + p(p-1)/2! . \Delta^2 v_{-1} +\Delta^2 v_o /2 + p(p-1)[p – ½] /3! *\Delta^3 v_{-1}$ $+ p(p-1)(p+1)(p-2)/4! *\Delta^4 v_{-2} + \Delta^4 v_{-1} /2 +.....$

Then simplifying this ;

$V_p = (1-p) v_o + [p(p-1) /4 + p(p-1) [p – ½]/6 * \Delta^2 v_o]+ pv_1 +$
$[p(p-1) /4 + p(p-1) [p – ½]/6 * \Delta^2 v_o]* \Delta^2 v_o$

$= qv_o + q*(q^2 – 1^2) /3! \Delta^2 v_o + pv_1 + p(p^2-1^2) /3! \Delta^2 v_o$

CHAPTER – 3

INTERPOLATION WITH UNEQUAL INTERVALS

3.1. INTRODUCTION

In this chapter, we are going to find Interpolation with unequal intervals. If the values of u's are given at unequal intervals, our Newton's forward, backward and central difference interpolation formula will not be true. Therefore we bring a new idea of divided differences.

3.2. DIVIDED DIFFERENCES

Consider the function v=h(u) . Let the values $h(u_0),h(u_1),\ldots,h(u_m)$ corresponding to the arguments $u_0,u_1,\ldots,u_m$ respectively, where the intervals $u_1-u_0,u_2-u_1,\ldots,u_m-u_{m-1}$ need not be equal.

The first divided difference of h(u) for u_0,u_1 is denoted by $h(u_0,u_1)$ or $[u_0,u_1]$.

$$h(u_0,u_1)=[u_0,u_1]=\frac{h(u_1)-h(u_0)}{u_1-u_0}$$

$$h(u_1,u_2)=\frac{h(u_2)-h(u_1)}{u_2-u_1}$$

$$h(u_{m-1},u_m)=\frac{h(u\_m)-h(u_{m-1})}{u_m-u_{m-1}}\ ,m=1,2,\ldots,m$$

The second divided difference of h(u) for u_0,u_1,u_2 is

$$h(u_0,u_1,u_2)=\frac{h(u_1,u_2)-h(u_0,u_1)}{u_2-u_0}$$

The third divided difference of h(u) for u_0,u_1,u_2,u_3 is

$$h(u_0,u_1,u_2,u_3)=\frac{h(u_1,u_2,u_3)-h(u_0,u_1,u_2)}{u_3-u_0}$$

Argument	Entry	First divided difference	Second divided difference	Third divided difference
u_0	$h(u_0)$	$h(u_0,u_1)$		
u_1	$h(u_1)$	$h(u_1,u_2)$	$h(u_0,u_1,u_2)$	
u_2	$h(u_2)$	$h(u_2,u_3)$	$h(u_1,u_2,u_3)$	$h(u_0,u_1,u_2,u_3)$
u_3	$h(u_3)$	$h(u_3,u_4)$	$h(u_2,u_3,u_4)$	$h(u_1,u_2,u_3,u_4)$
u_4	$h(u_4)$			

Example 3.1 : Find the divided difference of $h(u)=u^3+u^2+2$ for the arguments 2,4,7,12.

Solution :

u	h(u)	First div diff.	Second div diff.	Third div diff.
2	4	$\frac{70-12}{4-2} = 29$	$\frac{94-29}{7-2} = 13$	
4	70			
7	352			$\frac{23-13}{12-2} = 1$
12	1742	$\frac{352-70}{7-4} = 94$	$\frac{278-94}{12-4} = 23$	
		$\frac{1742-352}{12-7} = 278$		

3.3. PROPERTIES OF DIVIDED DIFFERENCE

Property 1 : The value of any divided difference does not depend on the order of the argument. Divided difference are symmetrical.

$h(u_0,u_1) = \frac{h(u_1)-h(u_0)}{u_1-u_0} = \frac{h(u_0)-h(u_1)}{u_0-u_1} = h(u_1,u_0) \qquad \dots(1)$

$h(u_0,u_1) = \frac{h(u_0)}{u_0-u_1} - \frac{h(u_1)}{u_0-u_1} = \frac{h(u_0)}{u_0-u_1} + \frac{h(u_1)}{u_1-u_0} \qquad \dots(2)$

Similarly,

$h(u_1,u_0) = \frac{h(u_1)}{u_1-u_0} + \frac{h(u0)}{u_0-u_1} \qquad \dots(3)$

From (2) and (3), we get,

$h(u_0,u_1) = h(u_1,u_0)$

Similarly,

$$h(u_0,u_1,u_2) = \frac{h(u_1,u_2)-h(u_0,u_1)}{u_2-u_0}$$

$$= \frac{1}{u_2-u_0}\left[\left[\frac{h(u_1)}{u_1-u_2} + \frac{h(u_2)}{u_2-u_1}\right] - \left[\frac{h(u_0)}{u_0-u_1} + \frac{h(u_1)}{u_1-u_0}\right]\right]$$

$$= \frac{1}{u_2-u_0}\left[\left[\frac{1}{u_1-u_2} - \frac{1}{u_1-u_0}\right] h(u_1) + \frac{h(u_2)}{u_2-u_1} - \frac{h(u_0)}{u_0-u1}\right]$$

$$= \frac{h(u_0)}{(u_0-u_1)(u_0-u_2)} + \frac{h(u_1)}{(u_1-u_0)(u_1-u_2)} + \frac{h(u_2)}{(u_2-u_0)(u_2-u_1)} \qquad \dots(4)$$

From (4), we have , $h(u_0,u_1,u_2)= h(u_1,u_0,u_2)= h(u_1,u_2,u_0)= \dots$

By mathematical induction,we can show that

$$h(u_0,u_1,u_2,....,u_m)=\frac{h(u_0)}{(u_0-u_1)(u_0-u_2)...(u_0-u_m)}+\frac{h(u_1)}{(u_1-u_0)(u_1-u_2)...(u_1-u_m)}+\frac{h(u_2)}{(u_2-u_0)(u_2-u_1)..(u_2-u_m)}+\cdots+$$

$$:\quad \frac{h(u_m)}{(u_m-u_0)(u_m-u_1)...(u_m-u_{m-1})}$$

Therefore,w.r.t any two arguments,divided differences are symmetrical.

Property2 : Operator Δ is linear.

Proof :

let h(u) and p(u) be two functions and let γ and δ be two Constants , then

$$\Delta[\gamma h(u)+\delta p(u)=\frac{[[\gamma h(u_1)+\delta p(u_1)]-[\gamma h(u_0)+\delta p(u_0)]]}{u_1-u_0}$$

$$=\frac{\gamma(h(u_1)-h(u_0))}{u_1-u_0}+\frac{\delta(p(u_1)-p(u_0))}{u_1-u_0}$$

$$=\gamma\Delta h(u)+\delta\Delta p(u)$$

Property 3 : The m-th divided difference of an m-th digree polynomial is constant.

Proof :

Let h(u)= u^m, m is a positive integer

$$h(u_0,u_1)=\frac{h(u_1)-h(u_0)}{u_1-u_0}=\frac{u_1^m-u_0^m}{u_1-u_0}$$

$$=u_1^{m-1}+u_0u_1^{m-2}+u_0^2u_1^{m-3}+\cdots+u_0^{m-1}$$

= a polynomial of degree (m-1) and symmetrical in u_0, u_1 with leading coefficient 1.

$$H(u_0,u_1,u_2)=\frac{h(u_1,u_2)-h(u_0,u_1)}{u_2-u_0}$$

$$=\frac{(u_2^{m-1}+u_1u_2^{m-2}+\cdots+u_1^{m-1})-(u_0^{m-1}+u_1u_0^{m-2}+\cdots+u_1^{m-1})}{u_2-u_0}$$

$$=\frac{u_2^{m-1}-u_0^{m-1}}{u_2-u_0}+\frac{u_1(u_2^{m-2}-u_0^{m-2})}{u_2-u_0}+\cdots+\frac{u_1^{m-2}(u_2-u_0)}{u_2-u_0}$$

$$=(u_2^{m-2}+u_0u_2^{m-3}+\cdots+u_0^{m-2})+u_1(u_2^{m-3}+u_0u_2^{m-4}+\cdots+u_0^{m-3})+\cdots+u_1^{m-2}$$

=Polynomial of degree (m-2) and symmetrical in u_0, u_1, u_2 with leading coefficient 1.

Continuing this way,the t-th divided difference of u^m will be a polynomial of degree (m-r) and symmetrical in $u_0, u_1, ..., u_t$ with leading coefficient 1.

Hence the m-th divided difference of um will be a polynomial of degree 0 with leading coefficient 1.

That is ,$\Delta^m u^m$ =1

$\Delta^{m+j}u^m = 0$ for $j = 1,2,\ldots$

Hence,$\Delta^m(a_0u^m + a_1u^{m-1} + \cdots + a_m)$

= $a_0\Delta^m u^m + a_1\Delta^m u^{m-1} + \cdots + \Delta^m a_n$

= $a_0.1 + 0 + \cdots + 0 = a_0$

3.4. RELATION BETWEEN DIVIDED DIFFERENCE AND FORWARD DIFFERENCE.

If $u_0, u_1, u_2, \ldots.$ are equally spaced, then $u_1 - u_0 = u_2 - u_1 = u_3 - u_2 = \cdots = u_m - u_{m-1}$ =d

$$h(u_1, u_0) = \frac{h(u_1) - h(u_0)}{u_1 - u_0} = \frac{\Delta h(u_0)}{d}$$

$$h(u_0) = \frac{h(u_1) - h(u_0)}{u_2 - u_0} = \frac{\frac{1}{d}\Delta h(u_1) - \frac{1}{d}\Delta h(u_0)}{2d}$$

$= \frac{1}{2d^2}\Delta^2 h(u_0)$

Similarly , $h(u_0) = \frac{\Delta^3 h(u_0)}{3d^3}$

$h(u_0) = \frac{\Delta^m h(u_0)}{md^m}$

THEOREM : Newton’s interpolation formula for unequal intervals / Newton’s divided difference formula

Let v=h(u)

It takes the values $h(u_0), h(u_1), \ldots h(u_m)$ corresponding to $u_0, u_1, \ldots u_m$. We

$$\text{know that } h(u, u_0) = \frac{h(u) - h(u_0)}{u - u_0}$$

$$\text{Therefore, } h(u) = h(u_0) + (u - u_0)h(u, u_0) \qquad \ldots (1)$$

$$\text{similarly, } h(u, u_0, u_1) = \frac{h(u, u_0) - h(u_0, u_1)}{u - u_1}$$

$$\text{therefore, } h(u, u_0) = h(u_0, u_1) + (u - u_1)h(u, u_0, u_1)$$

substitute $h(u, u_0)$ in (1)

$$h(u) = h(u_0) + (u - u_0)h(u_0, u_1) + (u - u_0)(u - u_1)h(u, u_0, u_1) \qquad \ldots (2)$$

$$\text{Again } h(u, u_0, u_1, u_2) = \frac{h(u, u_0, u_1) - h(u_0, u_1, u_2)}{u - u_2}$$

therefore, $h(u, u_0, u_1) = h(u_0, u_1, u_2) + (u - u_2)h(u, u_0, u_1, u_2)$

substitute, $h(u, u_0, u_1)$ in (2)

$$h(u) = h(u_0) + (u - u_0)h(u_0, u_1) + (u - u_0)(u - u_1)h(u_0, u_1, u_2) + (u - u_0)(u - u_1)(u - u_2)h(u, u_0, u_1, u_2) \quad \ldots .(3)$$

ccontinuing in this way, we get

$$\begin{aligned} h(u) = h(u_0) &+ (u - u_0)h(u_0, u_1) + (u - u_0)(u - u_1)h(u_0, u_1, u_2) \\ &+ (u - u_0)(u - u_1)(u - u_2)h(u, u_0, u_1, u_2) + \cdots \\ &+ (u - u_0)(u - u_1) \ldots (u - u_{m-1})h(u_0, u_1, \ldots, u_m) \\ &+ (u - u_0)(u - u_1) \ldots (u - u_m)h(u_0, u_1, \ldots, u_m) \quad \ldots (4) \end{aligned}$$

If $h(u)$ is a polynomial of degree m, then $h(u, u_0, u_1, \ldots, u_m) = 0$

hence (4) becomes ,

$$\begin{aligned} h(u) = h(u_0) &+ (u - u_0)h(u_0, u_1) + (u - u_0)(u - u_1)h(u_0, u_1, u_2) + \cdots \\ &+ (u - u_0)(u - u_1) \ldots (u - u_{m-1})h(u_0, u_1, \ldots, u_m) \quad \ldots (5) \end{aligned}$$

(5)is called Newton's divided difference interpolation formula for unequal intervals.

Example 3.2 : From the following table, find h(u) and hence find h(5) using Newton's interpolation formula

u	2	3	8	9
h(u)	2	6	6	5

Solution :

u	h(u)	First div diff.	Second div diff.	Third div diff.
2	2	$\frac{6-2}{3-2} = 4$		
3	6			
8	6	$\frac{6-6}{8-3} = 0$	$\frac{0-4}{8-2} = -\frac{2}{3}$	$\frac{\left(-\frac{1}{6}\right) + \left(\frac{2}{3}\right)}{9-2} = \frac{1}{14}$
9	5	$\frac{(5-6)}{9-8} = -1$	$\frac{-1-0}{9-3} = -\frac{1}{6}$	

$$By\ Newton's\ interpolation\ formula\ for\ unequal\ interval,$$

$$h(u) = h(u_0) + (u - u_0)h(u_0, u_1) + (u - u_0)(u - u_1)h(u_0, u_1, u_2) + \cdots$$

$$= 2 + (u - 2)4 + (u - 2)(u - 3)\left(-\frac{2}{3}\right) + (u - 2)(u - 3)(u - 8)\left(\frac{1}{14}\right)$$

$$= \frac{1}{42}[\ 2 + 4u - 8 + (u^2 - 5u + 6)(-28) + (u^2 - 5u + 6)(u - 8)(3)]$$

$$= \frac{1}{42}[4u - 6 + (-28u^2 + 140u - 168) + (u^3 - 5u^2 + 6u - 8u^2 + 40u - 48)]$$

$$= \frac{1}{42}[4u - 6 + (-28u^2 + 140u - 168) + (3u^3 - 39u^2 + 138u - 144)]$$

$$h(u) = \frac{1}{42}[3u^3 - 67u^2 + 282u - 318]$$

$$h(5) = \frac{1}{42}[372 - 1675 + 1410 - 318]$$

$$= \frac{1}{42}[1782 - 1993] = -\frac{211}{42}$$

= 5.02381

3.5. LAGRANGE 'S INTERPOLATION FORMULA [FOR UNEQUAL INTERVALS]

When the values of independent variable are not equally spaced ,also when the difference between the dependent variable are not small, we use Lagrange's interpolation formula.

Let v=h(u) be a function such that h(u) has values $v_0, v_1, v_2, \dots, v_m$ corresponding to $u = u_0, u_1, u_2, \dots, u_m$.

There are (m+1) paired values $(u_j, v_j), j = 0,1,2, \dots, m.$

Therefore,h(u) can be represented as a polynomial of degree m in u.

Let

$$\begin{aligned} h(u) = a_0(u-u_1)(u-u_2)\dots(u-u_m) + a_1(u-u_0)(u-u_2)(u-u_3)\dots(u-u_m) \\ + a_2(u-u_0)(u-u_1)(u-u_3)\dots(u-u_m) + \cdots \\ + a_j(u-u_0)(u-u_1)\dots(u-u_{j-1})(u-u_{j+1})..(u-u_m) + \cdots \\ + a_m(u-u_0)(u-u_1)\dots(u-u_{m-1}) \qquad \dots(1) \end{aligned}$$

It is true for all values of u.

sub $u = u_0$ and $v = v_0$ in (1), we get,6

$$v_0 = a_0(u_0 - u_1)(u_0 - u_2)\dots(u_0 - u_m)$$

$$Therefore\ , a_0 = \frac{v_0}{(u_0 - u_1)(u_0 - u_2)\dots(u_0 - u_m)}$$

$$similarly, let\ u = u_1, v = v_1, we\ get$$

$$a_1 = \frac{v_1}{(u_1 - u_0)(u_1 - u_2)(u_1 - u_3)\dots(u_1 - u_m)}$$

Continuing this way ,we get

$$a_m = \frac{v_m}{(u_m - u_0)(u_m - u_1)\dots(u_m - u_{m-1})}$$

Substitute this values in (1), we have,

$$v = h(u) = \frac{(u-u_1)(u-u_2)\dots(u-u_m)}{(u_0-u_1)(u_0-u_2)\dots(u_0-u_m)}.v_0 + \frac{(u-u_0)(u-u_2)\dots(u-u_m)}{(u_1-u_0)(u_1-u_2)\dots(u_1-u_m)}.v_1 + \cdots$$
$$+ \frac{(u-u_0)(u-u_1)\dots(u-u_{j-1})(u-u_{j+1})\dots(u-u_m)}{(u_j-u_0)(u_j-u_1)\dots(u_j-u_{j-1})(u_j-u_{j+1})\dots(u_j-u_m)}.v_j + \cdots$$
$$+ \frac{(u-u_0)(u-u_1)\dots.(u-u_{m-1})}{(u_m-u_0)(u_m-u_2)\dots(u_m-u_{m-1})}.v_m \qquad \dots\dots(2)$$

(2) is called Lagrange's interpolation formula for unequal intervals.

Example 3.3 : Using Lagrange's interpolation formula , find h(10.5) from the given table.

u	8	9	10	11
h(u)	4	2	2	10

Solution :

By Lagrange's formula,

$$v = h(u) = \frac{(u-9)(u-10)(u-11)}{(8-9)(8-10)(8-11)} \times 4 + \frac{(u-8)(u-10)(u-11)}{(9-8)(9-10)(9-11)} \times 2$$
$$+ \frac{(u-8)(u-9)(u-11)}{(10-8)(10-9)(10-11)} \times 2 + \frac{(u-8)(u-9)(u-10)}{(11-8)(11-9)(11-10)} \times 10$$

$$h(10.5) = \frac{(1.5)(0.5)(-0.5)}{(-1)(-2)(-3)} \times 4 + \frac{(2.5)(0.5)(-0.5)}{(1)(-1)(-2)} \times 2 + \frac{(2.5)(1.5)(-0.5)}{(2)(1)(-1)} \times 2 + \frac{(2.5)(1.5)(0.5)}{(3)(2)(1)} \times 10$$

=0.25-0.625+1.875+3.125

=4.625

4. SPLINE INTERPOLATION

we have so far discussed to find m-th order polynomial passing through (n+1) given data points using various methods.

It is important to note that while these methods can be useful in finding an approximate function that passes through the given data points, they may not always be accurate or suitable for all situations. Therefore, it is important to carefully consider the underlying assumptions and limitations of each method before applying them to real-world problems.

In spline interpolation, the interpolating function is represented by a piecewise polynomial function, where each polynomial is defined on a subinterval of the given data points. These polynomials are typically chosen to be of low degree, such as cubic or quadratic, and are smooth at the points where they meet, ensuring that the overall interpolating function is also smooth.

There are various types of spline used in interpolation, including natural splines, clamped splines, and periodic splines, each with their own set of properties and characteristics. The choice of which spline to use depends on the specific requirements of the problem at hand.

4.1 LINEAR SPLINES

Let the given data points be $\left(u_j, v_j\right)$,

$$j = 0,1,2, \dots m \tag{1}$$

Where, $a = u_0 < u_1 < u_2 < \cdots < u_n = b$

and let $d_j = u_j - u_{j-1}$, $\quad j = 1,2, \dots, m$

Further, let $s_j(u)$ be the spline of degree one defined in the interval $\left[u_{j-1}, u_j\right]$. Obviously, $s_j(u)$ represents a straight line joining the points $\left(u_{j-1}, v_{j-1}\right)$ and (u_j, v_j). Hence, we write

$$s_j(u) = v_{i-1} + k_j\left(u - u_{j-1}\right) \tag{2}$$

Where,

Slope, $k_j = \frac{v_j - v_{j-1}}{u_j - u_{j-1}}$

Where, j = 1, 2 ,...,m

we obtain different splines of degree one valid in the subintervals I to m, respectively. It is clear that $s_j(u)$ is continuous at both the end points.

Example:4.1.1

Given the set of data points (1,-8) (2,-1) and (3,18) satisfying the function v = h(u), find the linear splines satisfying the given data. Determine the approximate values of v(2.5) and v'(2).

>> Let the given points be P(1, - 8) Q(2, - 1) and R(3,18)

Equation of PQ is

$$s_1(u) = -8 + (u-1)7$$

$$= 7u - 15$$

and equation of QR is

$$s_2(u) = -1 + (u-2)19$$

$$= 19u - 39$$

Since u = 2.5 belongs to the interval [2, 3], we have

$v(2.5) \approx s_2(2.5) = 19(2.5) - 39$

$= 8.5$

And

$v'(2.0) \approx k_1 = 19$

It is easy to check that the splines $s_j(u)$are continuous in [1, 3] but their slopes are discontinuous. This is clearly a drawback of linear splines.

4.2 QUADRATIC SPLINES

With reference to the data points given linear splines (1) such that, (u_j, v_j) $, where\ j = 1,2,\dots,m$. Let $s_j(u)$ be the quadratic spline approximating the function $v = h(u)$ in the interval $[u_{j-1}, u_j]$, where

$u_j - u_{j-1} = d_j$.

Let $s_j(u)$ and $s_j'(u)$ be continuous in $[u_0, u_m]$ and let

$$s_j(u_j) = v_j, \qquad i = 0,1,2,\dots,m \tag{3}$$

Since $s_j(u)$ is a quadratic in $[u_{j-1}, u_j]$, it follows that $s_j'(u)$ is a linear function and therefore we write

$$s_j'(u) = \frac{1}{d_j}\left[(u_j - u)m_{i-1} + (u - u_{j-1})k_j\right], \quad (4)$$

Where, $k_j = s_j'(u_j)$

Integrating the equation $s_j'(u)$ with respect to u, we obtain

$$s_j(u) = \frac{1}{d_j}\left[-\frac{(u_j-u)^2}{2}\, k_{j-1} + \frac{(u-u_{j-1})^2}{2} k_j\right] + c_j \quad (5)$$

where c_j are constants to be determined. Substitute $u = u_{j-1}$, then we get

$$c_j = v_{j-1} + \frac{1}{d_j} \times \frac{d_j^2}{2} k_{j-1} = v_{j-1} + \frac{d_j}{2} k_{j-1}$$

Hence we get,

$$s_j(u) = \frac{1}{d_j}\left[-\frac{(u_j - u)^2}{2} k_{j-1} + \frac{(u - u_{j-1})^2}{2} k_j\right] + v_{j-1} + \frac{d_j}{2} k_{j-1} \quad (6)$$

Still here the value of k_j is unknown. To determine the k_j, we use the condition of continuity of the function since the first derivatives are already continuous. For the continuity of the function $s_j(u)$ at $u = u_j$, we must have

$$s_j(u_j -) = s_{j+1}(u_j +)$$

Therefore we obtain,

$$s_j(u_j -) = \frac{d_j}{2} k_j + v_{j-1} + \frac{d_j}{2} k_{j-1} = \frac{d_j}{2}(k_{j-1} + k_j) + v_{j-1} \quad (7)$$

Further,

$$s_{j+1}(u) = \frac{1}{d_{j+1}}\left[-\frac{(u_{j+1} - u)^2}{2} k_j + \frac{(u - u_j)^2}{2} k_{j+1}\right] + v_j + \frac{d_{j+1}}{2} k_j$$

and therefore

$$s_{j+1}(u_j +) = -\frac{d_{j+1}}{2} k_j + v_j + \frac{d_{j+1}}{2} k_j$$

$$= v_j \tag{8}$$

Equating $s_j(u_j -)$ and $s_{j+1}(u_j +)$, we get the recurrence relation

$$k_{j-1} + k_j = \frac{2}{d_j}(v_j - v_{j-1}), \tag{9}$$

$$j = 1,2, \dots, m$$

for the spline first derivatives k_j, the above equation constitute m equations in (m+1) unknowns, viz, $k_0, k_1, \dots, k_m$. Hence, we require one more condition to determine the k_j uniquely. There are several ways of choosing this condition. One natural way is to choose $s_1''(u_1) = 0$, since the mechanical spline straightens out in the end intervals. Such a spline is called a natural spline. Differentiating equation (6) twice with respect to u, we obtain

$$s_j''(u) = \frac{1}{d_j}(-k_{j-1} + k_j)$$

Or.

$$s_1''(u_1) = \frac{1}{d_1}(k_1 - k_0)$$

Hence, we have the additional condition as

$$k_0 = k_1 \tag{10}$$

Therefore, equations. (9) and (10) can be solved for k_j, which when substituted in (6) gives the required quadratic spline.

4.3. CUBIC SPLINES

We consider the same set of data points, viz., the data defined in (4.1), and let $s_j(u)$ be the cubic spline defined in the intervel $[u_{j-1}, u_j]$. There are some conditions for the natural cubic splines such as

(i) $s_j(u)$ is almost a cubic in each subinterval $[u_{j-1}, u_j]$, $j = 1,2, \dots, m$

(ii) $s_j(u) = v_j$, $j = 0,1,2, \dots, m$

(iii) $s_j(u), s_j'(u)$, and $s_j''(u)$ are continuous in $[u_0, u_m]$ and

(iv) $s_j''(u_0) = s_j''(u_m) = 0$

To derive the governing equations of the cubic spline, we observe that the spline second derivatives must be linear.

Hence, we have in $[u_{j-1}, u_j]$:

$$s_j"(u) = \frac{1}{d_j}\left[(u_j - u)K_{j-1} + (u - u_{i-1})K_j\right] \tag{11}$$

Where $k_j = u_j - u_{j-1}$ and $s_j"(u_j) = K_j$, for all i. Obviously, the spline second derivatives are continuous. Integrating the above equation twice with respect to u, we get

$$s_j(u) = \frac{1}{d_j}\left[\frac{(u_j - u)^3}{6}K_{j-1} + \frac{(u - u_{j-1})^3}{6}K_j\right] + c_j(u_j - u) + b_j(u - u_{j-1}) \tag{12}$$

where c_j and b_j are constants to be determined.

Using conditions $s_j(u_{j-1}) = v_{j-1}$ and $s_j(u_j) = v_j$, we immediately obtain

$c_j = \frac{1}{d_j}\left(v_{j-1} - \frac{d_j^2}{6}K_{i-1}\right)$ and $b_j = \frac{1}{d_j}\left(v_j - \frac{d_j^2}{6}K_j\right)$

Substituting for c_j and b_j in (4.12), we obtain

$$\begin{aligned} s_j(u) = \frac{1}{d_j}[&\frac{(u_j - u)^3}{6}K_{j-1} + \frac{(u - u_{j-1})^3}{6}K_j \\ &+ \left(v_{j-1} - \frac{d_j^2}{6}K_{j-1}\right)(u_j - u) \\ &+ \left(v_j - \frac{d_j^2}{6}K_j\right)(u - u_{j-1}) \end{aligned} \tag{13}$$

In the above equation, the spline second derivatives, k_j are still unknown. To determine them, we use the condition of continuity of $s_j'(u)$. From (13), we obtain by differentiation:

$$s_j'(u) = \frac{1}{d_j}\left[-\frac{3(u_j - u)^2}{6}K_{j-1} + \frac{3(u - u_{j-1})^2}{6}K_j - \left(v_{j-1} - \frac{d_j^2}{6}K_{j-1}\right) + \left(v_j - \frac{d_j^2}{6}K_j\right)\right]$$

Setting $u = u_j$ in the above, we obtain the left-hand derivatives

$$s_j{}'(u_j-) = \frac{d_j}{2}K_j - \frac{1}{d_j}\left(v_{j-1} - \frac{d_j^2}{6}K_{j-1}\right) + \frac{1}{h_j}\left(v_j - \frac{d_j^2}{6}K_j\right)$$

$$= \frac{1}{d_j}\left(v_j - v_{j-1}\right) + \frac{d_j}{6}K_{j-1} + \frac{d_j}{3}K_j \qquad (14)$$

$$(j = 1,2,\dots,m)$$

To obtain the right-hand derivative, we need first to write down the equation of the cubic spline in the subinterval (u_j, u_{j+1}). We do this by setting $j = j+1$ in equation (13)

$$s_{j+1}(u) = \frac{1}{h_{j+1}}\left[\frac{(u_{j+1}-u)^3}{6}K_j + \frac{(u-u_j)^3}{6}K_{j+1} + \left(v_j - \frac{d_{j+1}^2}{6}K_j\right)(u_{j+1}-u) + \left(v_{j+1} - \frac{h_{j+1}^2}{6}K_{j+1}\right)(u-u_j)\right] \qquad (15)$$

where $d_{j+1} = u_{j+1} - u_j$. Differentiating the above equation and setting $u = u_j$, we obtain the right-hand derivative at $u = u_j$

$$s_{j+1}{}'(u_j+) = \frac{1}{h_{j+1}}\left(v_{j+1} - v_j\right) - \frac{d_{j+1}}{3}K_j - \frac{d_{j+1}}{6}K_{j+1} \qquad (16)$$

$$(j = 0,1,\dots,m-1)$$

Equating the equations (4.14) and (4.16), we get the recurrence relation

$$\frac{d_j}{6}K_{j-1} + \frac{1}{3}\left(d_j + d_{j+1}\right)M_j + \frac{d_{j+1}}{6}K_{j+1} = \frac{v_{j+1} - v_j}{d_{j+1}} - \frac{v_j - v_{j-1}}{d_j} \qquad (17)$$

$$(j = 1,2,\dots,m-1)$$

For equal intervals, we have $d_j = d_{j+1} = d$ and Equation (17) simplifies to

$$K_{j+1} + 4K_j + K_{j+1} = \frac{6}{d^2}\left(v_{j+1} - 2v_j + v_{j-1}\right) \qquad (18)$$

$$(i = 1,2,\dots,m-1)$$

The system of the above equation has some special significance. If K_0 and K_m ae known, then the system can be written as

$$2(d_1 + d_2)K_1 + d_2K_2 = 6\left(\frac{v_2 - v_1}{d_2} - \frac{v_1 - v_0}{d_1}\right) - d_1K_0$$

$$d_2K_1 + 2(d_2 + d_3)K_2 + d_3M_3 = 6\left(\frac{v_3 - v_2}{d_3} - \frac{v_2 - v_1}{d_2}\right)$$

$$d_3K_2 + 2(d_3 + d_4)K_3 + d_4K_4 = 6\left(\frac{v_4 - v_3}{h_4} - \frac{v_3 - v_2}{d_3}\right)$$

:

$$d_{m-1}K_{m-2} + 2(d_{m-1} + d_m)K_{m-1} = 6\left(\frac{v_m - v_{m-1}}{d_m} - \frac{v_{m-1} - v_{m-2}}{d_{m-1}}\right) - d_m K_m \quad (19)$$

Equations (17) or (18) constitute a system of (m - 1) equations and with the two conditions in (iv) for the natural spline, we have a complete system which can be solved for the K_j. Systems of the above form (19) are called tridiagonal systems. When the Kj are known, equation (13) then gives the required cubic spline in the subinterval $[u_{j-1}, u_j]$. Also, the v_j' can be obtained from equations (14) and (16).

4.3.1 MINIMIZING PROPERTY OF CUBIC SPLINES

We prove this property for the natural cubic spline. Let $s(u)$ be the natural cubic spline interpolating the set of data points $(u_j, v_j), j = 0,1,2, \dots, m$ where it is assumed that $a = u_0 < u_1 < u_2 < \cdots < u_n = b$. Since s(u) is the natural cubic spline, we have $s(u_j)$for all j and also $s"(u_0) = s"(u_m) = 0$.

Let $w(u)$be a function such that $w(u_j) = v_j$ for all j, and $w(u), w'(u), w"(u)$ are continuous in [a, b] Then the integral defined by

$$I = \int_a^b [w"(u)]^2 du$$

will be minimum if and only if $w(u) = s(u)$. This means that s(u) is the smoothest function interpolating to the set of data points defined above, since the second derivative is a good approximation to the curvature of a curve, We write

$$\int_a^b [w"(u)]^2 du = \int_a^b [s"(u)+w''(u)-s''(u)]^2 du$$

$$= \int_a^b [s"(u)^2 du + 2\int_a^b s"(u)[w(u)-s"(u)]du$$

$$+ \int_a^b [w''(u) - s''(u)]^2 du \quad (20)$$

Now, $\int_a^b s"(u)[w"(u)-s"(u)]du = \sum \int_{uj}^{uj+1} s(u)[w(u)-s(u)]du$

$$= \sum \{s"(u)[w'(u) - s'(u)]\}$$

$$-\sum \int_{uj}^{uj+1} s'''(u)[w'(u) - s'(u)]du \quad (21)$$

The first term in above equation simplifies to

$$s''(u_m)[w'(u_m)-s'(u_m)]-s''(u_0)[w'(u_0)-s'(u_0)]$$

Since $s''(u_n) = s''(u_0) = 0$, the above expression vanishes. Similarly, the second term in (21) is zero since s'''(u) has a constant value in each interval and $s(u_j) = w(u_j) = v_j$ for all j. Hence, (20) becomes

$$\int_a^b \left[w''(u)\right]^2 du = \int_a^b \left[s''(u)\right]^2 du + \int_a^b [w''(u)- s''(u)]^2 du \qquad (22)$$

or

$$\int_a^b [w''(u)]^2 du \geq \int_a^b [s''(u)]^2 du$$

It follows that the integral

$$I = \int_a^b [w''(u)]^2 du$$

will be minimum if and only if

$$\int_a^b [w''(u)- s''(u)]^2 du = 0 \qquad (23)$$

which means that $w''(u)= s''(u)$. Hence $w(u) - s(u)$ is a polynomial in u of degree at most three in [a, b]. But the difference $w(u) - s(u)$ vanishes at the points j= 0,1,2,...,m. It therefore follows that

$$w(u) = s(u), \qquad a \leq u \leq b$$

4.3.2. ERROR IN THE CUBIC SPLINE AND ITS DERIVATIVES

An estimation of error in the cubic spline and its derivatives is important for assessing the accuracy and reliability of the interpolating function. One way to estimate the error in the cubic spline is to use the error bounds derived from the theory of approximation.

The following theorem provides an estimate of the error in the natural cubic spline:

Theorem :

If $v\ c^2[a,b], a = u_0 < u_1 < u_2 < \cdots < u_m = b$ and if $s(u)$ is the natural cubic spline for which

$$s(u_j) = v_j, \qquad i = 0,1,2,\dots,m$$

Then

$$\max|v(u) - s(u)| \le \frac{1}{2}Kd^2 \tag{24}$$

Where

$$d = u_{j+1} - u_j, \qquad j = 0,1,2,\dots,m$$

And

$$M = \max|v''(u)|, \qquad u_0 \le u \le u_m$$

As the interval length d becomes smaller, the natural cubic spline interpolant provides a better approximation to the underlying function. This is because the cubic spline interpolant is designed to be smooth and continuous, which makes it a more accurate representation of a smooth function than other interpolation methods like Lagrange interpolation, which can exhibit oscillations and overshoots between data points.

To estimate the error in the first derivatives, we use the recurrence relation,

$$k_{j-1} + 4k_j + k_{j+1} = \frac{3}{d}(v_{j+1} - v_{j-1})$$

That is,

$$s'(u_{j-1}) + 4s'(u_j) + s'(u_{j+1}) = \frac{3}{d}(v_{j+1} - v_{j-1})$$

This equation can rewrite by using the operator notation,

$$(E^{-1}+4+E)\, s'(u_j) = \frac{3}{h}(E - E^{-1})v_j$$

Since $E=e^{dD}$, where D= d/du, the above equation becomes

$$(e^{-dD}+4+ e^{dD})\, s'(u_j) = \frac{3}{d}(e^{dD} - e^{-dD})v_j \tag{25}$$

now

$e^{dD} = 1 + dD + \frac{d^2D^2}{2!} + \frac{d^3D^3}{3!} + \frac{d^4D^4}{4!} + \frac{d^5D^5}{5!} + \cdots$

and

$e^{-dD} = 1 - dD + \frac{d^2D^2}{2!} - \frac{d^3D^3}{3!} + \frac{d^4D^4}{4!} - \frac{d^5D^5}{5!} + \cdots$

hence

$e^{dD} + e^{-dD} = 2\left(1 + \frac{d^2D^2}{2} + \frac{d^4D^4}{24} + \frac{d^6D^6}{720} + \cdots\right)$

and

$e^{dD} - e^{-dD} = 2\left(dD + \frac{d^3D^3}{6} + \frac{d^5D^5}{120} + \cdots\right)$

using the above expression (25), we get

$$\left[2\left(1 + \frac{d^2D^2}{2} + \frac{d^4D^4}{24} + \cdots\right) + 4\right] s'(u_j) = \frac{3}{d} \times 2\left(dD + \frac{d^3D^3}{6} + \frac{d^5D^5}{120} + \cdots\right) v_j$$

$$= 6\left(D + \frac{d^2D^3}{6} + \frac{d^4D^5}{120} + \cdots\right) v_j$$

This implies

$$s'(u_j) = \frac{6\left(D + \frac{d^2D^3}{6} + \frac{d^4D^5}{120} + \cdots\right)}{6 + d^2D^2 + \frac{d^4D^4}{12} + \cdots} v_j$$

$$= \frac{D + \frac{d^2D^3}{6} + \frac{d^4D^5}{120} + \cdots}{1 + \frac{d^2D^2}{6} + \frac{d^4D^4}{72} + \cdots} v_j$$

$$= \left(D + \frac{d^2D^3}{6} + \frac{d^4D^5}{120} + \cdots\right) \left[1 + \left(\frac{d^2D^2}{6} + \frac{d^4D^4}{72} + \cdots\right)\right]^{-1} v_j$$

$$= \left(D + \frac{d^2D^3}{6} + \frac{d^4D^5}{120} + \cdots\right)\left[1 - \left(\frac{d^2D^2}{6} + \frac{d^4D^4}{72} + \cdots\right) + \left(\frac{d^2D^2}{6} + \frac{d^4D^4}{72} + \cdots\right)^2 - \ldots\right] v_j$$

$$= \left(D + \frac{d^2D^3}{6} + \frac{d^4D^5}{120} + \cdots\right)\left(1 - \frac{d^2D^2}{6} - \frac{d^4D^4}{72} - \cdots + \frac{d^4D^4}{36} + \cdots\right) v_j$$

$$= \left(D + \frac{d^2D^3}{6} + \frac{d^4D^5}{120} + \cdots\right)\left(1 - \frac{d^2D^2}{6} + \frac{d^4D^4}{72} - \cdots\right) v_j$$

$$= \left(D - \frac{d^2 D^3}{6} + \frac{d^4 D^5}{72} - \cdots + \frac{d^2 D^3}{6} - \frac{d^4 D^5}{36} + \frac{d^4 D^5}{120} +. \ldots \right) v_j$$

$$= \left(D - \frac{1}{180} d^4 D^5 + \cdots \right) v_j$$

Hence

$$s'(u_j) = v_j' - \tfrac{1}{180} d^4 v_j{}^{(5)} O(d^6) \qquad (26)$$

Similarly we can derive the equation:

$$s''(\mathrm{u_j}) = \mathrm{v}''(u_j) - \tfrac{1}{12} d^2 v^{(4)}(u_j) + \tfrac{1}{360} d^4 v^{(6)}(u_j) + O(d^6)$$

$$\tfrac{1}{2}\left[s'''(u_j +) + s'''(u_j -)\right] = v'''(u_j) + \tfrac{1}{12} d^2 v^{(5)}(u_j) + O(d^4)$$

$$s'''(u_j +) - s'''(u_j -) = d v^{(4)}(v_j) - \tfrac{1}{720} d^5 v^{(8)}(u_j) + O(d^7) \qquad (27)$$

From the equations (26) and (27), we get

$$v'(u_j) = s'(u_j) + O(d^4)$$

$$v''(\mathrm{u_j}) = \mathrm{s}''(u_j) + \tfrac{1}{12} d^2 v^{(4)}(u_j) + O(d^4)$$

$$v'''^{(u_j)} = \frac{1}{2}\left[s'''(u_j +) + s'''(u_j -)\right] + O(d^2)$$

$$v^{(4)}(u_j) = \tfrac{1}{2}\left[s'''^{(u_j+)} - s'''^{(u_j-)}\right] + O(d^4)$$

The above relations demonstrate that $v'''^{(u_j)}$ is approximately less accurate than $v'^{(u_j)}, v''^{(u_j)} and\ v^{(4)}(u_i)$.

CONCLUSION

it will examine the original pixel and then create a function that will describe the new pixel to create a new data set. Interpolation is a statistical method where unknown values are found using a known set of values.

There are different types of interpolation such as linear interpolation,spline interpolation,cubic interpolation etc.

There are many applications of interpolation ,some of which are :

1.Spline interpolation is often used in numerical analysis and computer graphics to represent curves and surfaces. It has several advantages over other interpolation methods, such as polynomial interpolation, including reduced oscillations, improved accuracy, and better preservation of derivatives.

2.Scientists may need to use a computer to do complicated calculations and it may take very long time. In such cases,they can use interpolation to convert their calculation to a slightly less complicated version. This will take less time and energy.

3.When an image is made larger,new pixels must be created instead of the existing pixels which cannot be stretched. Interpolation is used in the software that enlarges these pixels. First of all, the software spreads out the existing pixels, leaving many gaps and spaces,into a new image size. Then

REFERENCES

- S.S. Sastry: Introductory methods of numerical analysis (fourth edition).
- S.S. Sastry: Introductory methods of numerical analysis (fifth edition).
- Anthony Ralston, Philip Rabinowitz: A first course in numerical analysis (second edition).
- Dr. P. Kandasamy, Dr. K. Thilagavathy, Dr. K. Gunavathi: Numerical methods.