A STUDY ON GREEN'S FUNCTIONS

"A STUDY ON GREEN'S FUNCTION"

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DECLARATION

We hereby declare that this project entitled "**A STUDY ON GREEN'S FUNCTION**" is a bonafide record of work done by us under the supervision ofMs. KARTHIKA V, Department of Mathematics, Bharata Mata College, Thrikkakara and the work has not previously formed by the basis for the award of any academic qualification, fellowship or other similar title of any other University or Board.

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This is to certify that the project entitled **"A STUDY ON GREEN'S FUNCTION"** submitted jointly by AISWARYA P.U, THEJUSLAL K.S, AMALA GEORGE, DENILJOHN M.S, ATHIRA K.B in partial fulfillment of the requirements for the B.Sc. Degree in mathematics is a bonafide record of the studies undertaken by them under my supervision at the Department of Mathematics, Bharata Mata College, Thrikkakara, during 2021 – 2022. This dissertation has not been submitted for any other degree elsewhere.

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ABSTRACT

This project is about Green’s functions and its solutions. Green’s functions, a mathematical function that was introduced by George Green in 1793 to 1841. Green’s functions are used for solving Ordinary and Partial Differential Equations in different dimensions and for time-dependent and time-independent problem,and also in physics and mechanics, specifically in quantum field theory, electrodynamics and statistical field theory, to refer to various types of correlation functions

In chapter one we are going to discuss about what is green's function and it's techniques.We are also discussing about its Operator and forms of Green's function.

In chapter two we are talking about initial value problem involving nonhomogeneous differential equation using Green's function.

In chapter three is discussing about the boundary value Green's function and how to solve nonhomogeneous boundary value problems.This chapter is also discussing about the propertiesof green's function.

In last chapter four we are going to examine nonhomogeneous heat equation which play an important role in finding solution of a nonhomogeneous partial differential equation.

CHAPTER 1

INTRODUCTION

Green's functions are named after the British mathematician George Green, who developed the concept in the 1830s. Green's function methods enable the solution of a differential equation containing an inhomogeneous term (often called a source term) to be related to an integral operator. It can be used to solve both partial and exact differential equations.

George Green (14 July 1793 to 31 May 1841) was a British mathematical physicist who wrote an Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism'. The essay introduced several important concepts, among them a theorem similar to the modern Green's theorem, the idea of potential functions as currently used in physicsand the concept of what are now called Green's functions.

Green was the first person to create a mathematical theory of electricity and magnetism and his theory formed the foundation for the work of other scientists such as James Clerk Maxwell, William Thomson, and others.

Simple homogeneous differential equation
Consider the differential equation

$$\frac{d^2y}{dx^2} = 0$$

This can be solved very easily and we will get the solution as

Y =Ax + B

which is the equation for a straight line. The constants can be found if boundary conditions are given. Similarly consider another homogeneous equation

$$\frac{d^2y}{dx^2} + k^2\text{y} = 0$$

This can be solved to get

y = A sin k x + B cos k x

Thus there are simple techniques available to solve homogeneous equations. But if we replace them with source terms like

$$\frac{d^2y}{dx^2} = \ln x$$

$$\frac{d^2y}{dx^2} + k^2 y = \tan x$$

then the problem become difficult to solve. Before thinking of solving such nonhomogeneous equations let us look at different types of differential operators.

Sturm Liouville operator

Sturm Liouville operator is the most general form of second order differential operator which can be written in the equation form as

$$\mathcal{L}y = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = 0$$

For

$$\frac{d^2y}{dx^2} = 0$$

$$p(x) = 1 \; and \; q(x) = 0 \; and \; for$$

$$\frac{d^2y}{dx^2} + k^2 y = 0$$

$$p(x) = 1 \; and \; q(x) = k^2.$$

Any differential operator can be changed into SL operator form.

Dirac delta function

While studying GF techniques we will encounter some properties of Dirac delta function. They are

$$\int_{allspace} \delta(x-t)dx = 1$$

$$\int \delta(x-t)f(t)dt = f(x)$$

Greens function technique

Suppose SL operator operating on a function y(x) gives as

$$\mathcal{L}y(x) = f(x)(1)$$

which is a nonhomogeneous equation. To solve this NHE let us define

$\mathcal{L}G(x,t) = \delta(x-t)(2)$

so that we can show that if we define y(x) = $\int G(x,t)f(t)dt$ we will get equation (1). The proof of this argument is given below

Proof:

$$\mathcal{L}y(x) = \mathcal{L}\int G(x,t)f(t)dt$$

Interchanging integral and differential= $\int \mathcal{L}G(x,t)f(x)dt$

Using the definition of Greens function = $\int \delta(x-t)f(t)dt$

And using the property Dirac delta function that

$\int \delta(x-t)f(t)dt$ = f(x) we get $\mathcal{L}y(x) = f(x)$

Definition

Generally speaking, a Green's function is an integral kernel that can be used to solve differential equations from a large number of families including simpler examples such as ordinary differential equations with initial or boundary value conditions, as well as more difficult examples such as inhomogeneous partial differential equations (PDE) with boundary conditions.

One dimensional Greens function and its properties:

So let us start with $\mathcal{L}G(x,t) = \delta(x-t)$

Taking the SL operator

$$\frac{d}{dx}\left(p(x)\frac{d}{dx}G(x,t)\right)+q(x)G(x,t)=\delta(x-t)$$

Integrating over x for a small interval t - $\varepsilon\ to\ t+\varepsilon$

$$\int_{t-\varepsilon}^{t+\varepsilon}\frac{d}{dx}\left(p(x)\frac{d}{dx}G(x,t)\right)dx+\int_{t-\varepsilon}^{t+\varepsilon}q(x)G(x,t)dx=\int_{t-\varepsilon}^{t-\varepsilon}\delta(x-t)dx$$

ε is a small quantity. Hence RHS is 1. Taking the second part as zero

$$p(t+\varepsilon)\frac{dG}{dx_{t+\varepsilon}}-p(t-\varepsilon)\frac{dG}{dx_{t-\varepsilon}}=1$$

In the limit $\varepsilon\to 0$

$$p(t)\left(\frac{dG}{dx_{t+\varepsilon}}-\frac{dG}{dx_{t-\varepsilon}}\right)=1$$

$$\frac{dG}{dx_{t+\varepsilon}}-\frac{dG}{dx_{t-\varepsilon}}=\frac{1}{p(t)}$$

$$\frac{dG_2}{dx}-\frac{dG_1}{dx}=\frac{1}{p(t)}$$

This property shows that the values of GF must be different for x less than t and x greater than t. So let label GF before t as$G_1(x,t)$ and GF after t as $G_2(x,t)$. We had taken the second integral as zero which means that

$$G_2(x,t+\varepsilon)-G_1(x,t-\varepsilon)=0$$

At x = t$G_1=G_2$ or Greens function is

1. Continuous at boundary and
2. Derivative of the Greens function is discontinuous.

These are the two properties of one dimensional Green's function.

Form of Greens function

Next is to find G_1 and G_2 . Assume

$$G_1(x,t) = C_1 u_1(x)$$

And

$$G_2(x,t) = C_2 u_2(x)$$

where C_1 and C_2 which are functions of t are to be determined. The Greens functions are determined using the two properties we got. The continuity of Greens function demands that

$$C_2 u_2(t) - C_1 u_1(t) = 0$$

Discontinuity of Greens function demands that

$$C_2 u_2'(t) - C_1 u_1'(t) = \frac{1}{p(t)}$$

Multiplying the first equation by $u_2'(t)$ and second by $u_2(t)$

$$C_2 u_2'(t) u_2(t) - C_1 u_2'(t) u_1(t) = 0$$

$$C_2 u_2'(t) u_2(t) - C_1 u_1'(t) u_2(t) = -\frac{1}{p(t)} u_2(t)$$

Subtracting gives

$$C_1 u_1'(t) u_2(t) - C_1 u_2'(t) u_1(t) = \frac{u_2(t)}{p(t)}$$

$$C_1(u_2 u_1' - u_1 u_2') = \frac{u_2(t)}{p(t)}$$

If W = $u_1 u_2' - u_2 u_1', (is\ also\ called\ Wronskian\ ) Then$

$$C_1 = \frac{u_2(t)}{Wp(t)}$$

$$C_2 = \frac{u_1(t)}{Wp(t)}$$

Hence G_1(x,t) $= \frac{u_1(x) u_2(t)}{Wp(t)}$

G2(x,t) $=\frac{u_2(x)u_1(t)}{Wp(t)}$

Then we get the solution as

$$y(x) = \int_a^t G_1(x,t)f(t)dt + \int_t^b G_2(x,t)f(t)dt .$$

CHAPTER 2

Initial Value Green's Functions

In this chapter we will investigate the solution of initial value problems involving nonhomogeneous differential equations using Green's functions. Our goal is to solve the nonhomogeneous differential equation

a(t)y'' + p(t)y'(t) + c(t)y(t)= f(t)(1)

subject to the initial conditions

$$y(0) = y_0 \quad y'(0) = \upsilon_0$$

Since we are interested in initial value problems, we will denote the independent variable as a time variable, t.

Equation (1) can be written compactly as

$$L[y] = f',$$

where L is the differential operator

$$L = a(t)\frac{d^2}{dt^2} + b(t)\frac{d}{dt} + c(t).$$

The solution is formally given by

$$y = L^{-1}[f]$$

The inverse of a differential operator is an integral operator, which we seek to write in the form

$y(t) = \int G(t, \tau) f(\tau)\, d\tau.$

The function $G(t, \tau)$ is referred to as the kernel of the integral operator and is called the Green's function.

From (1) using the Method of Variation of Parameters. Letting,

$y_p(t) = c_1(t)\, y_1(t) + c_2(t) y_2(t)$ (2)

we found that we have to solve the system of equation

$c_1'(t)y_1(t) + c_2'(t)y_2(t) = 0$

$c_1'(t)\, y_1'(t) + c_2'(t) y_2'(t) = \frac{f(t)}{g(t)}$ (3)

This system is easily solved to give

$$c_1'(t) = - \frac{f(t)y_2(t)}{a(t)[y_1(t)y_2'(t) - y_1'(t)y_2(t)]}$$

$$c_2'(t) = \frac{f(t)y_1(t)}{a(t)[y_1(t)y_2'(t) - y_1'(t)y_2(t)]} \quad (4)$$

We note that the denominator in these expressions involves the Wronskian of the solutions to the homogeneous problem, which is given by the determinant

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

When $y_1(t)$ and $y_2(t)$ are linearly independent, then the Wronskian is not zero and we are guaranteed a solution to the above system.
So, after an integration, we find the parameters as

$$c_1(t) = -\int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)} d\tau$$

$$c_2(t) = \int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)} d\tau, \quad (5)$$

where t_0 and t_1 are arbitrary constants to be determined from the initial conditions.Therefore, the particular solution of (1) can be written as

$$y_p(t) = y_2(t) \int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)} d\tau - y_1(t) \int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)} d\tau \quad (6)$$

We begin with the particular solution (6) of the nonhomogeneous differential equation (1). This can be combined with the general solution of the homogeneous problem to give the general solution of the nonhomogeneous differential equation:

$$y_p(t) = c_1 y_1(t) + c_2 y_2(t) + y_2(t) \int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)} d\tau - y_1(t) \int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)} d\tau \quad (7)$$

However, an appropriate choice of t_0 and t_1 can be found so that we need not explicitly write out the solution to the homogeneous problem,$c_1 y_1(t) + c_2 y_2(t)$. However, setting up the solution in this form will allow us to use $t_0$and $t_1$to determine particular solutions which satisfies certain homogeneous conditions. In particular, we will show that Equation (7) can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \int_0^t G(t,\tau) f(\tau)\, d\tau , \qquad (8)$$

where the function G(t, τ) will be identified as the Green's function.
The goal is to develop the Green's function technique to solve the initial value problem

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t) , \quad y(0) = y_0 , \quad y'(0) = \upsilon_0 \quad (9)$$

We first note that we can solve this initial value problem by solving two separate initial value problems. We assume that the solution of the homogeneous problem satisfies the original initial conditions:

$$a(t)y_h''(t) + b(t)y_h'(t) + c(t)y_h(t) = 0 , \; y_h(0) = y_0 , \; y_h'(0) = \upsilon_0 \qquad (10)$$

We then assume that the particular solution satisfies the problem

$$a(t)y_p''(t) + b(t)y_p'(t) + c(t)y_p(t) = f(t) , \; y_p(0) = 0 , \; y_p'(0) = 0 \quad (11)$$

Since the differential equation is linear, then we know that y(t) = y_h(t) + y_p(t) is a solution of the nonhomogeneous equation. Also, this solution satisfies the initial conditions:

$$y(0) = y_h(0) + y_p(0) = y_0 + 0 = y_0 ,$$

$$y'(0) = y_h'(0) + y_p'(0) = \upsilon_0 + 0 = \upsilon_0$$

Therefore, we need only focus on finding a particular solution that satisfies homogeneous initial conditions. This will be done by finding values for t_0 and t_1 in Equation (6) which satisfy the homogeneous initial conditions,
$y_p(0) = 0$ and $y_p'(0) = 0$.

First, we consider $y_p(0) = 0$, We have

$$y_p(0) = y_2(0) \int_{t_1}^{0} \frac{f(\tau) y_1(\tau)}{a(\tau) W(\tau)} d\tau - y_1(0) \int_{t_0}^{0} \frac{f(\tau) y_2(\tau)}{a(\tau) W(\tau)} d\tau \quad (12)$$

Here, y_1(t) and y_2(t) are taken to be any solutions of the homogeneous differential equation. Let's assume that $y_1(0) = 0$ and $y_2(0) \neq 0$. Then, we have

$y_p(0) = y_2(0)\int_{t_1}^{0} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}d\tau$ (13)

We can force $y_p(0) = 0$ if we set $t_1 = 0$.

Now, we consider $y_p'(0) = 0$. First we differentiate the solution and find that

$$y_p'(t) = y_2'(t)\int_{0}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}d\tau - y_1'(t)\int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}d\tau, \quad (14)$$

since the contributions from differentiating the integrals will cancel. Evaluating this result at t = 0, we have

$$y_p'(0) = -y_1'(0)\int_{t_0}^{0} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}d\tau \quad (15)$$

Assuming that $y_1'(0) \neq 0$, we can set $t_0 = 0$. Thus, we have found that

$$y_p(x) = y_2(t)\int_{0}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}d\tau - y_1(t)\int_{0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}d\tau$$

$$= \int_{0}^{t}\left[\frac{y_1(\tau)y_2(\tau) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)}\right] f(\tau)d\tau \quad (16)$$

This result is in the correct form and we can identify the temporal, or initial value, Green's function. So, the particular solution is given as

$$y_p(t) = \int_{0}^{t} G(t,\tau) f(\tau)\, d\tau \quad (17)$$

where the initial value Green's function is defined as

$$G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)}$$

We summarize

Solution of IVP Using the Green's Function

The solution of the initial value problem,

$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t)$, $y(0) = y_0$, $y'(0) = \mho_0$, takes the form

$$y(t) = y_h(t) + \int_{0}^{t} G(t,\tau) f(\tau)\, d\tau \quad (18)$$

Where $G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)}$ (19)

is the Green's function and y_1, y_2, y_h are solutions of the homogeneous equation satisfying

$y_1(0) = 0, y_2(0) \neq 0, y_1'(0) \neq 0, y_2'(0) = 0, y_h(0) = 0, y_h'(0) = v_0$

Example 1. Solve the forced oscillator problem

x'' + x = 2 cost, x(0) = 4, x' (0) = 0.

Wefirst solve the homogeneous problem with nonhomogeneous initial conditions:

$x_h'' + x_h = 0$, $x_h(0) = 4$,x_h' (0) = 0.

The solution is easily seen to be $x_h(t) = 4 \cos t$.
Next, we construct the Green's function. We need two linearly independent solutions, y_1 (x), y_2 (x), to the homogeneous differential equation satisfying different homogeneous conditions, y_1 (0) =0 and $y_2'(0)$ =0. The simplest solutions are y_1(t)= sin t and y_2(t)=cos t. The Wronskian is found as

W(t) = $y_1(t)y_2'(t)$- $y_1'(t)$ y_2(t) = $-sin^2 t - cos^2 t$ = -1

Since a(t) = 1 in this problem, we compute the Green's function,

$$G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)}$$

= sintcosτ - sinτ $cost$

=sin(t -τ)   (20)

Note that the Green's function depends on t $-\tau$. While this is useful in some contexts, we will use the expanded form when carrying out the integration.We can now determine the particular solution of the nonhomogeneous differential equation.we have

$$x_p(t) = \int_0^t G(t,\tau) f(\tau)\, d\tau$$

$$= \int_0^t (\sin t \cos \tau - \sin \tau \cos t)(2 \cos\tau)\, d\tau$$

$$= 2 \sin t \int_0^t cos^2 \tau d\tau - 2\cos t \int_0^t sin\, \tau \cos\tau\, d\tau$$

$$= 2 \sin t\left[\frac{\tau}{2} + \frac{1}{2}\, sin 2\tau\right]_0^t - 2\cos t\left[\frac{1}{2} sin^2 \tau\right]_0^t$$

=t sin t   (21)

Therefore, the solution of the nonhomogeneous problem is the sum of the solution of the homogeneous problem and this particular solution: $x(t)= 4 \cos t + t \sin t$.

CHAPTER 3

Boundary Value Green's Functions

Wesolvednonhomogeneousinitialvalue problem in chapter 2 using a Green's function. In this chapter we will extend this method to the solution of nonhomogeneous boundary value problems using a boundary value Green's function. Recall that the goal is to solve the nonhomogeneous differential equation

L[y] = f, a ≤x ≤ b, where L is a differential operator and y(x) satisfies boundary conditions at x = a and x = b. The solution is formally given by

$$y = L^{-1}[f]$$

The inverse of a differential operator is an integral operator, which we seek to write in the form

$$y(x) = \int_a^b G(x,\zeta)\, f(\zeta) d\zeta$$

The function G(x, ζ) is referred to as the kernel of the integral operator and is called the Green's function. We will consider boundary value problems in Sturm-Liouville form,

$$\frac{d}{dx}\left(p(x)\,\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x)\ , \quad a < x < b \qquad (22)$$

with fixed values of y(x) at the boundary, y(a)= 0 and y(b) = 0. However, the general theory works for other forms of homogeneous boundary conditions.

We seek the Green's function by first solving the nonhomogeneous differential equation using the Method of Variation of Parameters.

We assume a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)\ ,$$

which is formed from two linearly independent solution of the homogeneous problem, $y_i(x)$, i = 1, 2. We had found that the coefficient functions satisfy the equations

$c_1'(x)\, y_1(x) + c_2'(x) y_2(x) = 0$

$c_1'(x)\, y_1'(x) + c_2'(x) y_2'(x) = \frac{f(x)}{p(x)}$ (23)

Solving this system we obtain

$c_1'(x) = -\frac{f y_2}{pW(y_1, y_2)}$,

$c_1'(x) = \frac{f y_1}{pW(y_1, y_2)}$,

where W $(y_1, y_2) = y_1 y_2' - y_1' y_2$ is the Wronskian. Integrating these forms and inserting the results back into the particular solution, we find

$$y(x) = y_2(x) \int_{x_1}^{x} \frac{f(\zeta) y_1(\zeta)}{p(\zeta) W(\zeta)} d\zeta - y_1(x) \int_{x_0}^{x} \frac{f(\zeta) y_2(\zeta)}{p(\zeta) W(\zeta)} d\zeta,$$

where x_0 and x_1 are to be determined using the boundary values. In particular, we will seek x_0 and x_1 so that the solution to the boundary value problem can be written as a single integral involving a Green's function. Note that we can absorb the solution to the homogeneous problem, $y_h(x)$, into the integrals with an appropriate choice of limits on the integrals.

We now look to satisfy the conditions y(a) = 0 and y(b) = 0. First we use solutions of the homogeneous differential equation that satisfy $y_1(a) = 0$, $y_2(b) = 0$ and $y_1(b) \neq 0$, $y_2(a) \neq 0$. Evaluating y(x) at x = 0, we have

$$y(a) = y_2(a) \int_{x_1}^{a} \frac{f(\zeta) y_1(\zeta)}{p(\zeta) W(\zeta)} d\zeta - y_1(a) \int_{x_0}^{a} \frac{f(\zeta) y_2(\zeta)}{p(\zeta) W(\zeta)} d\zeta,$$

$$= y_2(a) \int_{x_1}^{a} \frac{f(\zeta) y_1(\zeta)}{p(\zeta) W(\zeta)} d\zeta \quad (24)$$

We can satisfy the condition at x = a if we choose x_1= a. Similarly, at x = b we find that

$$y(b) = y_2(b) \int_{x_1}^{b} \frac{f(\zeta) y_1(\zeta)}{p(\zeta) W(\zeta)} d\zeta - y_1(b) \int_{x_0}^{b} \frac{f(\zeta) y_2(\zeta)}{p(\zeta) W(\zeta)} d\zeta,$$

$$= -y_1(b) \int_{x_0}^{b} \frac{f(\zeta) y_2(\zeta)}{p(\zeta) W(\zeta)} d\zeta \quad (25)$$

This expression vanishes for x_0 = b.So, we have found that the solution takes the form

$$y(x) = y_2(x)\int_a^x \frac{f(\zeta)y_1(\zeta)}{p(\zeta)W(\zeta)}d\zeta - y_1(x)\int_b^x \frac{f(\zeta)y_2(\zeta)}{p(\zeta)W(\zeta)}d\zeta \quad (26)$$

This solution can be written in a compact form just like we had done for the initial value problem in chapter 2. We seek a Green's function so that the solution can be written as a single integral. We can move the functions of x under the integral. Also, since a < x < b, we can flip the limits in the second integral. This gives

$$y(x) = \int_a^x \frac{f(\zeta)y_1(\zeta)y_2(x)}{p(\zeta)W(\zeta)}d\zeta + \int_x^b \frac{f(\zeta)y_1(x)y_2(\zeta)}{p(\zeta)W(\zeta)}d\zeta \quad (27)$$

This result can now be written in a compact form:

Boundary Value Green's Function

The solution of the boundary value problem

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x), \quad a < x < b,$$

$$y(a) = 0, \; y(b) = 0 \quad (28)$$

takes the form

$$y(x) = \int_a^b G(x,\zeta)\, f(\zeta)d\zeta \quad (29)$$

where the Green's function is the piecewise defined function

$$G(x,\zeta) = \begin{cases} \frac{y_1(\zeta)y_2(x)}{pW}, & a \le \zeta \le x, \\ \frac{y_1(x)y_2(\zeta)}{pW}, & x \le \zeta \le b, \end{cases} \quad (30)$$

Where $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous problem satisfying $y_1(a) = 0$, $y_2(b) = 0$ and $y_1(b) \neq 0$, $y_2(a) \neq 0$.

The Green's function satisfies several properties, which we will explore later. For example, the Green's function satisfies the boundary conditions at x = a and x= b. Thus,

$$G(a,\zeta) = \frac{y_1(a)y_2(\zeta)}{pW} = 0$$

$$G(b,\zeta) = \frac{y_1(\zeta)y_2(b)}{pW} = 0$$

Also, the Green's function is symmetric in its arguments. Interchanging the arguments gives

$$G(\zeta,x) = \begin{cases} \frac{y_1(x)y_2(\zeta)}{pW}, a \le x \le \zeta, \\ \frac{y_1(\zeta)y_2(x)}{pW}, \zeta \le x \le b, \end{cases} \quad (31)$$

But a careful look at the original form shows that

$$G(x,\zeta) = G(\zeta,x)$$

We will make use of these properties in the next section to quickly determine the Green's functions for other boundary value problems.

Example 2. Solve the boundary value problem y'' =x^2,y(0) = 0 = y(l) using the boundary value Green's function.

We first solve the homogeneous equation, y'' = 0. After two integrations, we have y(x) =Ax + B, for A and B constants to be determined. We need one solution satisfying y_1(0)= 0 Thus, 0= y_1(0) = B.

So, we can pick y_1(x) = x, since A is arbitrary.
The other solution has to satisfy y_2(1) = 0. So,

0= y_2(1) = A + B.

This can be solvedfor B = -A. Again, A is arbitrary and we will choose A = -1. Thus, y_2(x) =1 – x,

For this problem p(x)=1. Thus, for y_1(x)= x and y_2(x) = 1- x,

P(x)W(x)= $y_1(x)y_2'(x) - y_1'y_2(x)$=x(-1)-1(1-x) = -1.

Note that p(x)W(x) is a constant, as it should be.
Now we construct the Green's function. We have

$$G(x,\zeta)=\begin{cases} -\zeta(1-x), \ 0 \le \zeta \le x, \\ -x(1-\zeta), \ x \le \zeta \le 1, \end{cases} \quad (32)$$

Notice the symmetry between the two branches of the Green's function. Also, the Green's function satisfies homogeneous boundary conditions: $G(0,\zeta)$=0 , from the lower branch, and G(l, ζ) = 0, from the upper branch.

Finally, we insert the Green's function into the integral form of the solution and evaluate the integral.

$$y(x)= \int_0^1 G(x,\zeta)f(\zeta)d\zeta$$

$$=\int_0^1 G(x,\zeta)\zeta^2 d\zeta$$

$$=-\int_0^x \zeta(1-x)\zeta^2\,d\zeta - \int_x^1 x(1-\zeta)\zeta^2\,d\zeta$$

$$= -(1-x)\int_0^x \zeta^3\,d\zeta - x\int_x^1(\zeta^2-\zeta^3)d\zeta$$

$$=-(1-x)\left[\frac{\zeta^4}{4}\right]_0^x - x\left[\frac{\zeta^3}{3}- \quad \frac{\zeta^4}{4}\right]_x^1$$

$$=-\frac{1}{4}(1-x)x^4 - \frac{1}{12}x(4-3) + \frac{1}{12}x(4x^3-3x^4)$$

$$=\frac{1}{12}(x^4-x) \qquad (33)$$

Checking the answer, we can easily verify that y" = x^2, y(0) = 0, and y(l) = 0.

Properties of Green's Functions

In this section we will elaborate on some of these properties as a tool for quickly constructing Green's functions for boundary value problems. We list five basic properties:

1. **Differential Equation:**

The boundary value Green's function satisfies the differential equation

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\zeta)}{\partial x}\right) + q(x)G(x,\zeta) = 0, x \neq \zeta.$$

This is easily established. For x < ζ we are on the second branch and G(x, ζ) is proportional to y_1(x). Thus, since y_1(x) is a solution of the homogeneous equation, then so is G(x, ζ). For x >ζwe are on the first branch and G(x, ζ) is proportional to y_2(x). So, once again G(x, ζ) is a solution of the homogeneous problem.

2. Boundary Conditions:

In the example in the last section we had seen that G(a, ζ) =0 and G(b, ζ)=0. For example, for x= a we are on the second branch and G(x, ζ)is proportional to y_1(x). Thus, whatever condition y_1(x). satisfies, G(x, ζ) will satisfy. A similar statement can be made for x = b

3. Symmetry or Reciprocity: G(x, ζ) = G(ζ,x)

We had shown this reciprocity property in the last section.

4. Continuity of G at x =ζ: G (ζ^+ , ζ) = G(ζ^-, ζ)

Here we define $\zeta^\pm$ through the limits of a function as x approaches ζ from above or below. In particular,

$$G(\zeta^+, x) = \lim_{x \downarrow \zeta} G(x, \zeta), \; x > \zeta$$

$$G(\zeta^-, x) = \lim_{x \uparrow \zeta} G(x, \zeta), \; x < \zeta$$

Setting x= ζ in both branches, we have

$$\frac{y_1(\zeta) y_2(\zeta)}{pW} = \frac{y_1(\zeta) y_2(\zeta)}{pW}$$

Therefore, we have established the continuity of G(x, ζ) between the two branches at x = ζ.

5. Jump Discontinuity of $\frac{\partial G}{\partial x}$ at x = ζ:

$$\frac{\partial G(\zeta^+, \zeta)}{\partial x} - \frac{\partial G(\zeta^-, \zeta)}{\partial x} = \frac{1}{p(\zeta)}$$

This case is not as obvious. We first compute the derivatives by noting which branch is involved and then evaluate the derivatives and subtract them. Thus, we have

$$\frac{\partial G(\zeta^+, \zeta^-)}{\partial x} - \frac{\partial G(\zeta^-, \zeta)}{\partial x} = \frac{-1}{pW} y_1(\zeta) y_2'(\zeta) + \frac{1}{pW} y_1'(\zeta) y_2(\zeta)$$

$$=-\frac{y_1'(\zeta)y_2(\zeta)-y_1(\zeta)y_2'(\zeta)}{p(\zeta)(y_1(\zeta)y_2'(\zeta)-y_1'(\zeta)y_2(\zeta))}$$

$$=\frac{1}{p(\zeta)} \qquad (34)$$

Here is a summary of the properties of the boundary value Green's function based upon the previous solution.

Properties of the Green's Function

1. Differential Equation:

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\zeta)}{\partial x}\right)+q(x)G(x,\zeta)=0, x\neq\zeta.$$

2.Boundary Conditions: Whatever conditions y_1(x) and y_2(x) satisfy,G(x,ζ)will satisfy.

3.Symmetry or Reciprocity: G(x, ζ) = G(ζ,x)

4.Continuity of G at x = ζ: G (ζ^+ , ζ) = G(ζ^-, ζ)

5. Jump Discontinuity of $\frac{\partial G}{\partial x}$at x = ζ:

$$\frac{\partial G(\zeta^+,\zeta)}{\partial x}-\frac{\partial G(\zeta^-,\zeta)}{\partial x}=\frac{1}{p(\zeta)}$$

We now show how a knowledge of these properties allows one to quickly construct a Green's function with an example.

Example 3. Construct the Green's function for the problem

$$y'' + \omega^2 y = f(x), \qquad 0<x<1$$

$$y(0)=0=y(1)$$

With $\omega \neq 0$

1. Find solutions to the homogeneous equation.

General solution to the homogeneous equation is given as

$$y_h(x)=c_1 sin\omega x+ c_2 cos\omega x$$

Thus, for x $\neq \zeta$,

G(x, ζ) = $c_1(\zeta)\, sin\omega x + c_2(\zeta) cos\omega x$

2. Boundary Conditions

First we have G(0, ζ) = 0 for $0 \le x \le \zeta$. So,

G(0, ζ) = $c_2(\zeta) cos\omega x = 0$

So, G(x, ζ)= $c_1(\zeta)\, sin\omega x$, $0 \le x \le \zeta$

Second we have G(1, ζ) = 0 for $\zeta \le x \le 1$. So

G(1, ζ) = $c_1(\zeta)\, sin\omega + c_2(\zeta) cos\omega = 0$

A solution can be chosen with

$c_2(\zeta) = -c_1(\zeta) tan\omega$.

This gives G(x, , ζ) = $c_1(\zeta)\, sin\omega x - c_1(\zeta) tan\omega cos\omega x$

This can be simplified by factoring out the $c_1(\zeta)$ and placing the remaining terms over a common denominator. The result is

$$G(x, \zeta) = \frac{c_1(\zeta)}{cos\omega}[\sin\omega x cos\omega - sin\omega cos\omega x]$$
$$= -\frac{c_1(\zeta)}{cos\omega} sin\omega(1-x) \quad (35)$$

Since the coefficient is arbitrary at this point, as can

write the result as

G(x, ζ) = $d_1(\zeta)\sin\omega(1-x),\ \zeta \le x \le 1$

We note that we could have started with $y_2(x)$= $\sin\omega(1-x)$ as one of the linearly independent solutions of the homogeneous problem in anticipation that $y_2(x)$ satisfies the second boundary condition.

3. Symmetry or Reciprocity

We now impose that G(x, ζ) = G(ζ, x). To this point we have that

$$G(x, \zeta) = \begin{cases} c_1(\zeta)\sin\omega x, & 0 \le x \le \zeta \\ d_1(\zeta)\sin\omega(1-x), & \zeta \le x \le 1 \end{cases}$$

We can make the branches symmetric by picking the right forms for$c_1(\zeta)$ and $d_1(\zeta)$We choose $c_1(\zeta) = C sin\omega(1-\zeta)$ and $d_1(\zeta) = C sin\omega\zeta$ .Then,

$$G(x, \zeta) = \begin{cases} C\sin\omega(1-\zeta)\sin\omega x, & 0 \le x \le \zeta \\ C\sin\omega(1-x)\sin\omega\zeta, & \zeta \le x \le 1 \end{cases}$$

Now the Green's function is symmetric and we still have to determine the constant C. We note that we could have gotten to this point using the Method of Variation of Parameters result where C= $\frac{1}{pW}$

4.<u>Continuity of</u>G(x,ζ)

We already have continuity by virtue of the symmetry imposed in the last step.

5.<u>Jump Discontinuity in</u>$\frac{\partial}{\partial x}G(x,\zeta)$

We still need to determine C. We can do this using the jump discontinuity in the derivative:

$$\frac{\partial G(\zeta^+,\zeta)}{\partial x} - \frac{\partial G(\zeta^-,\zeta)}{\partial x} = \frac{1}{p(\zeta)}$$

For this problem p(x) =1. Inserting the Green's function, we have

$$1=\frac{\partial G(\zeta^+,\zeta)}{\partial x} - \frac{\partial G(\zeta^-,\zeta)}{\partial x}$$

$$=\frac{\partial}{\partial x}[Csin\omega(1-x)sin\omega\zeta]_{x=\zeta} - \frac{\partial}{\partial x}[Csin\omega(1-\zeta)sin\omega x]_{x=\zeta}$$

$$=-\omega Ccos\omega(1-\zeta)sin\omega\zeta - \omega Csin\omega(1-\zeta)cos\omega\zeta$$

$$=-\omega Csin\omega(\zeta+1-\zeta)$$

$$=-\omega Csin\omega \qquad (36)$$

Therefore,C= $-\frac{1}{\omega sin\omega}$

Finally, we have the Green's function:

$$G(x,\zeta)=\begin{cases}-\frac{\sin\omega(1-\zeta)\sin\omega x}{\omega sin\omega}, 0\le x\le\zeta\\ -\frac{\sin\omega(1-x)\sin\omega\zeta}{\omega sin\omega},\ \zeta\le x\le 1\end{cases} \qquad (37)$$

It is instructive to compare this result to the variation of parameters result.

Example 4. Use the Method of Variation of Parameters to solve

$$y'' + \omega^2 y = f(x),\ 0< x< 1,$$

$$y(0) = 0 = y(1),\ \omega \neq 0.$$

We have the functions $y_1(x)$= sinωx and $y_2(x)$= sin$\omega(1-x)$ as the solutions of the homogeneous equation satisfying $y_1(0) = 0$ and $y_2(1)$ = 0. We need to compute pW:

$$p(x)W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

$$= -\omega sin\omega x cos\omega(1-x) - \omega cos\omega x sin\omega(1-x)$$

$$= -\omega sin\omega \quad (38)$$

Inserting this result into the Variation of Parameters resultfor the Green's function leads to the same Green's function as above.

CHAPTER 4

The Nonhomogeneous Heat Equation

Boundary value of Green's function do not only arise in the Solution of nonhomogeneous ordinary differential equations. They are also important in arriving at the solution of nonhomogeneous partial differential equations. In this chapter we will show that this is the case by turning to the nonhomogeneous heat equation

Nonhomogeneous Time Independent Boundary Conditions

Consider the nonhomogeneous heat equation with nonhomogeneous boundary conditions:

$$u_t - ku_{xx} = h(x), 0 \leq x \leq L, t > 0$$
$$u(0,t) = a, u(L,t) = b,$$
$$u(x,0) = f(x). \quad (39)$$

We are interested in finding a particular solution to this initial-boundary value problem. In fact, we can represent the solution to the general nonhomogeneous heat equation as the sum of two solutions that solve different problems.

First, we let $v(x,t)$ satisfy the homogeneous problem

$$v_t - kv_{xx} = 0, 0 \leq x \leq L, t > 0,$$
$$v(0,t) = 0, v(L,t) = 0,$$
$$v(x,0) = g(x), \quad (40)$$

which has homogeneous boundary conditions.

Note: The steady state solution, $w(t)$ satisfies a nonhomogeneous differential equation with nonhomogeneous boundary conditions. The transient solution, $v(t)$, satisfies the homogeneous heat equation with homogeneous boundary conditions and satisfies a modified initial condition.

We will also need a steady state solution to the original problem. A steady state solution is one that satisfies $u_t = 0$. Let w(x) be the steady state solution. It satisfies the problems.

$$-kw_{xx} = h(x), 0 \leq x \leq L$$
$$w(0,t) = a, \; w(L,t) = b \quad (41)$$

Now consider $u(x,t) = w(x) + v(x,t)$, the sum of the steady state solution, $w(x)$, and the transient solution, $v(x,t)$ We first note that $u(x,t)$ satisfies the nonhomogeneous heat equation,

$$u_t - ku_{xx} = (w+v)_t - (w+v)_{xx}$$

$$= v_t - kv_{xx} - kw_{xx} \equiv h(x) \qquad (42)$$

The boundary conditions are also satisfied. Evaluating $u(x,t) at\ x = 0\ and\ x = L$, we have

$$u(0,t) = w(0) + v(0,t) = a,$$

$$u(L,t) = w(L) + v(L,t) = b. \qquad (43)$$

Note: The transient solution satisfies

$$v(x,0) = f(x) - w(x).$$

Finally, the initial condition gives

$$u(x,0) = w(x) + v(x,0) = w(x) + g(x).$$

Thusifwe set $u(x,t) = w(x) + v(x,t)$, will be the solution of the nonhomogeneous boundary value problem. We already know how to solve the homogeneous problem to obtain $v(x,t)$.So, we only need to find the steady state solution, $w(x)$.

There are several methods we could use to solve Equation $w(0,t) = a, w(L,t) = b$ for the steady state solution. One is the Method of Variation of Parameters, which is closely related to the Green's function method for boundary value problems which we described in chapter 3. However, we will just integrate the differential equation for the steady state solution directly to find the solution. From this solution we will be able to read off the Green's function.

Integrating the steady state equation,

$$\frac{dw}{dx} = -\frac{1}{k}\int_0^x h(z)dz + A,$$

where we have been careful to include the integration constant, $A = w'(0)$. Integrating again, we obtain

$$w(x) = -\frac{1}{k}\int_0^x \left(\int_0^y h(z)dz\right)dy + Ax + B,$$

here a second integration constant has been introduced. This gives the general solution for Equation.

The boundary conditions can now be used to determine the constants. It is clear that $B = a$ for the condition at $x = 0$ to be satisfied. The second condition gives

$$b = w(L) == -\frac{1}{k}\int_0^L \left(\int_0^y h(z)dz\right)dy + AL + a,$$

Solving for A, we have

$$A = -\frac{1}{kL}\int_0^L \left(\int_0^y h(z)dz\right)dy + \frac{b-a}{L}.$$

Inserting the integration constants, the solution of the boundary value problem for the steady state solution is then

$$w(x) = -\frac{1}{k}\int_0^x \left(\int_0^y h(z)dz\right)dy + \frac{x}{kL}\int_0^L \left(\int_0^y h(z)dz\right)dy + \frac{b-a}{L}x + a.$$

This is sufficient for an answer, but it can be written in a more compact form. In fact, we will show that the solution can be written in a way that a Green's function can be identified.

First, we rewrite the double integrals as single integrals. We can do this using integration by parts. Consider integral in the first term of the solution,

$$\mathrm{I} = \int_0^x \left(\int_0^y h(z)dz\right)dy,$$

Setting $u = \int_0^y h(z)dz \; and \; dv = dy$ in the standard integration by parts formula, we obtain

$$\begin{aligned}\mathrm{I} &= \int_0^x \left(\int_0^y h(z)dz\right)dy,\\ &= y\int_0^y h(z)dz - \int_0^x yh(y)dy\\ &= \int_0^x (x-y)h(y)dy \qquad (44)\end{aligned}$$

Thus, the double integral has now collapsed to a single integral. Replacing the integral in the solution, the steady state solution becomes

$$w(x) = -\frac{1}{k}\int_0^x (x-y)h(y)dy + \frac{x}{kL}\int_0^L (L-y)h(y)dy + \frac{b-a}{L}x + a.$$

We can make a further simplification by combining these integrals. This can be done if the integration range, $[0, L]$, in the second integral is split into two pieces, $[0, x] \; and \; [x, L]$ Writing the second integral as two integrals over these subintervals, we obtain

$$w(x) = -\frac{1}{k}\int_0^x (x-y)h(y)dy + \frac{x}{kL}\int_0^L (L-y)h(y)dy + \frac{x}{kL}\int_x^L (L-y)h(y)dy + \frac{b-a}{L}\,x + a. \qquad (45)$$

Next, we rewrite the integrals,

$$w(x) = -\frac{1}{k}\int_0^x \frac{L(x-y)}{L} h(y)dy$$
$$+\frac{1}{k}\int_0^x \frac{x(L-y)}{L} h(y)dy + \frac{1}{k}\int_x^L \frac{x(L-y)}{L} h(y)dy + \frac{b-a}{L}x + a. \quad (46)$$

It can now be seen how we can combine the first two integrals:

$$w(x) = -\frac{1}{k}\int_0^x \frac{y(L-x)}{L} h(y)dy + \frac{1}{k}\int_x^L \frac{x(L-y)}{L} h(y)dy + \frac{b-a}{L}x + a.$$

The resulting integrals now take on a similar form and this solution can be written compactly as

$$w(x) = -\int_0^L \mathrm{G(x,y)}[-\frac{1}{k}h(y)]dy + \frac{b-a}{L}x + a$$

Where

$$\mathrm{G(x,y)} = \begin{cases} \frac{x(L-y)}{L}, 0 \le x \le L, \\ \frac{y(L-x)}{L}, y \le x \le L, \end{cases}$$

so the Green's function for this problem.

The full solution to the original problem can be found by adding to this steady state solution a solution of the homogeneous problem,

$$u_t - ku_{xx} = 0, 0 \le x \le L, t > 0,$$
$$u(0.t) = 0, u(L,t) = 0,$$
$$u(x,0) = f(x) - w(x). \quad (47)$$

Time Dependent Boundary Conditions

When the boundary conditions are time dependent, we can also convert the problem to an auxiliary problem with homogeneous boundary conditions.

Consider the problem

$$u_t - ku_{xx} = h(x), 0 \le x \le L, t > 0,$$
$$u(0,t) = a(t), u(L,t) = b(t), t > 0$$
$$u(x,0) = f(x), 0 \le x \le L \quad (46)$$

We define $u(x,t) = v(x,t) + w(x,t)$, where $w(x,t)$ is a modified form of the steady state solution from the last section,

$$w(x,t) = a(t) + \frac{b(t) - a(t)}{L}x$$

Nothing that

$$u_t = v_t + a + \frac{b-a}{L}x,$$
$$u_{xx} = v_{xx}, \quad (47)$$

We find that $v(x,t)$ is a solution of the problem

$$v_t - kv_{xx} = h(x) - \left[a(t) + \frac{b(t)-a(t)}{L}x\right], 0 \le x \le L, t > 0,$$
$$v(0,t) = 0\,, v(L,t) = 0, t > 0,$$
$$v(x,0) = f(x) - \left[a(0) - \frac{b(0)-a(0)}{L}x\right], 0 \le x \le L \qquad (48)$$

Thus, we have converted the original problem into a nonhomogeneous heat equation with homogeneous boundary conditions and a new source term and new initial condition.

Example 5: Solve the problem

$$u_t - u_{xx} = x, 0 \le x \le 1, t > 0,$$
$$u(0,t) = 2, u(L,t) = t, t > 0$$
$$u(x,0) = 3sin2\pi x + 2(1-x), 0 \le x \le 1 \qquad (49)$$

We first define

$$u(x,t) = v(x,t) + 2 + (t-2)x$$

Then, v(x,t) satisfies the problem

$$v_t - v_{xx} = 0, 0 \le x \le 1, t > 0,$$
$$v(0,t) = 0, v(L,t) = 0, t > 0,$$
$$v(x,0) = 3sin\,2\pi x, 0 \le x \le 1 \quad (50)$$

This problem is easily solved. The general solution is given by

$$v(x,t) = \sum_{n=1}^{\infty} b_n sinn\pi x e^{-n^2\pi^2 t}$$

We can see that the Fourier coefficients all vanish except for b_2. This gives v(x,t) = 3sin $2\pi x e^{-4\pi 2t}$ and , therefore , we have found this solution

$$u(x,t) = 3sin2\pi x e^{-4\pi^2 t} + 2 + (t-2)x.$$

CONCLUSION

Green's function is an integral kernel that can be used to solve differential equations from a large number of families including simpler examples such as ordinary differential equations with initial or boundary value conditions, as well as more difficult examples such as inhomogeneous partial differential equations (PDE) with boundary conditions.

In Chapter1:We discussed about the definition of Green's function and we learnedabout it's operators and it's forms.

In chapter 2:We discussed about initial value Green's function.An initial condition is an extra bit of information about a differential equation that tells you the value of the function at a particular point. Differential equations with initial conditions are commonly called initial value problems involving nonhomogeneous differential equation using Green's function and also we examined their examples

In Chapter 3:We discussed about boundary value Green's function and how to solve nonhomogeneous boundary value problems.For a given boundary value problem, Green's function is a fundamental solution satisfying a boundary condition. One advantage of using Green's function is that it reduces the dimension of the problem by one.Here we also go through their examples.

In Chapter 4: Finally we discussed about non homogeneous heat equation.In this Chapter discussed about non homogeneous time independent boundary condition and nonhomogeneous time dependent boundary condition with examples.

Green function solutions are one of the most powerful analytical tools we have for solving partial differential equations, equations that arise in areas of physics such as electromagnetism (Maxwell’s equations), wave mechanics (elastic wave equation), optics (Helmholtz equation), quantum mechanics (Schrödinger and Dirac equations), fluid dynamics (fluid equations of motion), relativistic particle dynamics (Klein-Gordon equation) and general relativity (Einstein equations) to name but a few.

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