A GLIMPSE INTO THE PRELIMINARIES OF GAME THEORY

Report of the Project Submitted to
Mahatma Gandhi University Kottayam
in partial fulfillment of the requirement for the degree of
Bachelor of Science in Mathematics

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APRIL 2022

DECLARATION

We hereby declare that this dissertation entitled **"A GLIMPSE INTO THE PRELIMINARIES OF GAME THEORY"** is a bonafide work done by us under the supervision of **Dr. PAUL ISAAC** Associate Professor of Mathematics, Bharata Mata College, Thrikkakara and the work has not previously formed the basis for the award of any academic qualification, fellowship of the other similarity of any other University or Board.

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CERTIFICATE

This is to certify that this dissertation entitled **"A GLIMPSE INTO THE PRELIMINARIES OF GAME THEORY"** submitted by **AHSANA K SYED, AKASH P V, KOLLARA JOHN MATHEW, NIYA PHILIP and SANDRA KALADHARAN** in partial fulfillment of the requirements for the Bachelor's Degree in Mathematics is a bonafide record of the work undertaken by them under my supervision at Department of Mathematics, Bharata Mata College, Thrikkakara during 2019- 2022. This dissertation has not been submitted for any other degree elsewhere.

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ACKNOWELDGEMENT

With great pleasure and gratitude we acknowledge our indebtedness to those who have helped in completing this project. We take this opportunity to express our sincere thanks to all of them, although they all cannot be mentioned here.

We extend our heartfelt gratitude towards **Dr. Paul Isaac**, Associate Professor in Mathematics, Bharata Mata College, Thrikkakara for his valuable support and guidance that enabled us to make study in this topic and to prepare this dissertation.

We are also thankful to **Head of the Department and all other teachers in the Department** for their co-operation to prepare this report.

We also extend our sincere thanks to **the library staff and our friends** for their help and support during this study.

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CONTENTS

INTRODUCTION

A game, in the mathematical sense, is a situation in which players make rational decisions according to defined rules in an attempt to receive some sort of payoff. Game theory is the branch of mathematics which focuses on the analysis of such games.

In game theory, a player's strategy is any of the options which they choose in a setting where the outcome depends not only on their own actions but on the actions of others. The discipline mainly concerns the action of a player in a game affecting the behavior or actions of other players. A player's strategy will determine the action which the player will take at any stage of the game.

In this project, we would explore the fundamentals of Game Theory. The fundamentals includes notion of a zero-sum game, mixed and pure strategies and expected payoff. On this foundation, we build a player's optimal strategy for zero-sum games that is initiated by an analysis of the situation where the opponent's strategy is known. The crux of the dissertation is the maximin and minimax strategy which is also known as "the best of a set of worst possible outcomes or 1 payoffs" and "a tactic in which individuals attempt either to minimize their own maximum losses or to reduce the most an opponent will gain" respectively. The solution of the game and it's generalization to n options of a player is discussed towards the end. The project concludes with the applications of the techniques to the game theoretical analysis of the fishing strategies adopted by the members of Jamaican fishing village proposed by an anthropologist W.C.A. Davenport.

1. OPTIMAL RESPONSES TO SPECIFIC STRATEGIES

The fundamental notions of a zero-sum game, mixed and pure strategies, and expected payoff of are defined. The search for a player's optimal strategies for zero-sum games is initiated by an analysis of the situation where the opponent's strategy is known

It is time to make some formal definitions. For purely pedagogical reasons we begin with games in which each of the players has only two options

BOMBING SORTIES. Ruth and Charlie are generals of opposing armies. Everyday Ruth sends out a bombing sortie that consists of a heavily armed bomber plane and a lighter support plane. The sortie’s mission is to drop a single bomb on Charlie’s forces. However, a fighter plane of Charlie's is waiting for them in ambush and it will dive down and attack one of the planes in the sortie once. The bomber has an 80% chance of surviving such an attack, and if it survives it is sure to drop the bomb right on the target. General Ruth also has the option of placing the bomb on the support plane. In that case, due to this plane's lighter armament and lack of proper equipment, the bomb will reach its target with a probability of only 50% or 90%, depending on whether or not it is attacked Charlie’s fighter.

This information is summarized as mathematical representation of Bombing-sorties game array

80%	100%
90%	50%

Each 2 x2 zero-sum game has two players, we shall continue to call Ruth and Charlie. The mathematical analog of deciding on one of the options is the selection of either a row or a column of this array. Specifically, Ruth decides on a option by selecting a row of the array, whereas Charlie makes his decision by

specifying a column. Thus in Bombing-sorties Ruth's placement of the bomb in tbe bomber and Charie's attacking the support plane are tantamount to Ruth's selecting the first row and Charlie's selecting the second column of the array

80%	100%
90%	50%

Each individual play of the game consists of such a pair of selections, made simultaneously and independently. The selected row and column constitute the outcome of the play and its payoff of the entry of the array that is o contained in both of selections. Thus, the payoff of the play illustrated above is 100%. On the other hand, had Ruth selected the second row and had Charlie stayed with the second column the payoff would have been 50%. This payoff of course represents Ruth's winnings (and Charlie's loss) from that play and she will in general wish to maximize its value, whereas Charlie will be guided by the desire to minimize this payoff..

Informally speaking. a players' strategy is a decision on the frequency with which each available option will be chosen. More formally, a strategy is a pair of number [a,b], $0< a < 1$, $0< b < 1$, $a+b =1$

It was seen in Bombing-sorties that the specification of each player's strategy resulted in a situation where in an expected payoff could be computed.. This expected payoff can be defined and computed for arbitrary 2 x 2 games in a similar manner. Thus, given the strategies [1 –p,p] and [1-q,q] for general 2 x 2 zero-sum game

	1-q	q
1-p	a	b
p	c	d

the likelihood of Ruth getting the payoff a is the probability of her choosing the first row and Charlie choosing the first column. Since these choices are made independently, we conclude that the probability of Ruth getting payoff a is $(1-p)x(1-q)$.

Similarly,

The probability of Ruth getting payoff b is (1-p)xq
The probability of Ruth getting payoff c is px(1-q)
The probability of Ruth getting payoff d is pxq

As these four events are mutually exclusive and they exhaust all the possibilities it follows that Ruth's expected payoff is

(1-p) x (1-q) x a + (1-p) x q x b + p x (1-q) x c + p x q x d

Diagrams such as that of (1) above, wherein the players' strategies are appended to the game's array, are called auxiliary diagrams. They turn the computation of the expected payoff into a routine task and so will be repeatedly used in the sequel as a visual aid

EXAMPLE

For the Bombing-sorties game Ruth's strategy[. 3, .7] and Charlie's strategy [.6,.4] yield the auxiliary diagram

	0.6	0.4
0.3	80%	100%
0.7	90%	50%

The corresponding payoff is
.3 x .6 x 80% + .3 x .4 x 100% + .7 x .6 x 90% + .7 x .4 x50% = 14.4% + 12% +37.8% + 14% = 78.2%

In other words, when the players employ the above specified strategies, Ruth can expect 78.2% of the missions to be successful.

In Bombing-sorties we considered the question of what Charlie should do if he observes that Ruth bluffs by placing the bomb on the support plane 1/4 of the time. Let us reconsider this question in a somewhat more formal manner. Using the notation and terminology of the previous chapter, Ruth's decision to bluff in this manner is tantamount to adopting the strategy [.75,.25]and Charlie's search for an appropriate response reduces to finding a strategy [1-q,q] such that the corresponding expected payoff (rate of successful missions) is as low as possible this can now be made more methodical .The auxiliary diagram that describes this situation is

	1-q	q
0.75	80%	100%
0.25	90%	50%

And the expected payoff is

.75 x (1- q) x 80% + .75 x q x 100% + .25 x (1- q) x 90% + .25 x q x 50 % = .60(1- q) + .75q + .225(1- q) +.125q

= .60 - .60q + .75q + .225 - .225q +.125q = .825 + .05q

In other words, as a function of q, the expected payoff is .825 + .05q. bearing in mind that q is a probability and hence 0< q < 1, it follows that the expected payoff is least when q is 0, when it assumes the value

.825 + .05 x 0 = .825 = 82 .5 %

This means that Charlie's best response to Ruth's bluffing is to set q= 0 in his general strategy[1-q, q] . ie .he should ignore the bluffing and consistently attack the bomber

Suppose that instead of bluffing 1/4 of the time Ruth decides that she will place the bomb on the support plane 1/2 the time, that is, suppose that Ruth adopts the strategy[.5, .5]. What is Charlie's best strategy then? Now the auxiliary diagram is

	1-q	q
0.50	80%	100%
0.50	90%	50%

And the expected payoff is

.50 x (1-q) x80% + .05 x q x 100% + .05 x (1-q) x 90% + .05 x q x 50% = .04(1-q) + .05q + .45(1-q) + .25q

= .40-.04q + .05q + .45(1-q) + .25q

= .04- .04 q + .50q + .45 - .45q + .25q = .85 - .10q

Since Charlie is trying to minimize payoffs it is clearly to his advantage to assign to q the largest possible value, namely 1. This means that under these circumstances

Charlie's best strategy is [1 – 1,1] = [0, 1]. In other words, if Ruth places the bomb (at random) on either plane with frequency 1/2, then it is to Charlie's advantage to attack the weaker support plane consistently.

It will prove convenient to introduce some new terms here. In the face of any specific strategy of Ruth's, that strategy of Charlie's that results in the smallest

expected payoff to Ruth's called Charlie's optimal counterstrategy. Similarly, in the face of a specific strategy of Charlie's, that strategy of Ruth's that provides

her with the largest expected payoff is called her optimal counterstrategy. The conclusion of the above discussion is that when Ruth employs the mixed strategy[.75,.25,] Charlie's optimal counterstrategy is [1, 0] whereas when Ruth employs the strategy [.5,.5]. Charlie's optional counterstrategy is [0,1]. That both of the counterstrategies are pure is no coincidence and we formulate the general principle as a theorem .

THEOREM

If one player of a game employs a fixed strategy, then the opponent has an optimal counterstrategy that is pure.

This theorem reduces the task of determining a player's optimal counter strategy to his opponent's fixed strategy to a manageable number of computations. It should be pointed out that sometimes nonpure optimal counterstrategies are also available

THEOREM

In any 2 x 2 zero-sum game, if one player employs a fixed strategy, then the opponent has an optimal counterstrategy that is pure

PROOF :

Suppose E(p,q) is the expected payoff when Ruth and Charlie employ the strategies [1-p,p] and[1-q,q] respectively in the game

G =

a	b
c	d

$$E(p,q) = (1-p)(1-q)a + (1-p)qb + p(1-q)c + pqd$$

$$= p(-a+aq+c-bq-cq+dq) + (a-aq+bq).$$

If Charlie employs a fixed strategy [1-q, q] then the quantities a, b, c, d, q are all fixed and so E(p,q), as a function of p, $0 < p < 1$, assumes its minimum value at

$$P = \begin{cases} 0 & \text{if } -a + aq + c - bq - cq + d > 0 \\ 1 & \text{if } -a + aq + c - bq - cq + dq < 0 \end{cases}$$

In either case, Ruth has a pure optimal counterstrategy. The proof that Charlie has pure optimal counter strategy

$E(p,g) = (1- P)((1 - q)a + qb) + p((1 -q)c+ qd)$
$= (1 - p) E(0, q) + p E(1,q).$

Hence E(p,q), being a weighted average of E(0, q) and E(1,q), must lie between them. Consequently Ruth's optimal response to Charlie's [1 – q,q] can be obtained by setting p to be either 0 if $E(0,q) > E(1,q)$ or 1 if $E(1,q) > E(0,q)$.In other words, Ruth has a pure optimal counterstrategy.

2. The Maximin Strategy and The Minimax Strategy

a) The Maximin Strategy

The selection of an optimal strategy by each player without the knowledge of the competitor's strategy is the basic problem of playing games. The objective of the study is to know how these players select their respective strategy so that they may optimize their pay off. Such a decision-making criterion is referred to as the minimax- maximin principle.

A maximin strategy is a strategy in game theory where a player makes a decision that yields the 'best of the worst' outcome. All decisions will have costs and benefits, and a maximin strategy is one that seeks out the decision that yields the smallest loss.

a	b
c	d

In the previous chapter we have seen that for any strategy for Ruth there is a pure optimal counter strategy for Charlie. Let us consider the strategy of Ruth as q-1 and q. Then there are two possible pure strategy for Charlie viz [1,0] and [0,1] where the corresponding expected pay off is $r_1(p)$ and $r_2(p)$ respectively.

	1	0
1-p	A	b
p	C	d

$r_1(p)$

	0	1
1-p	a	b
p	c	d

$r_2(p)$

and we compute

$$r_1(p) = (1-p) \times 1 \times a + p \times 1 \times c$$
$$= a(1-p) + cp$$
$$= (c-a)\,p + a$$

$$r_2(p) = (1-p) \times 1 \times b + p \times 1 \times d$$
$$= b(1-p) + dp$$
$$= (d-b)\,p + b$$

If $E_R(p)$ denotes the expected pay off selected from $r_1(p)$ and $r_2(p)$ by Charlie, then, since Charlie can be relied on to lower this pay off as much as he can,

$E_R(p)$ = the lesser of $\{r_1(p), r_2(p)\}$.

This completely determines Ruth's expected pay off $E_R(p)$, as a function of p, i.e., as a function of her strategy. We shall use the graph of this function in order to suggest a strategy for Ruth.

Since the independent variable p appears in the expression of $r_1(p)$ and $r_2(p)$ with a degree at most 1, the graphs of these functions are straight lines.

In as much as p denotes a probability, we have $0 \leq p \leq 1$, and so these graphs consist of line segments that lie over the interval [0,1] on the p-axis. More specifically, since

$$r_1(0) = a\,(1-0) + c \times 0 = a$$
$$r_1(1) = a\,(1-1) + c \times 1 = c$$

It follows that the graph of $r_1(p)$ is the line segment joining the points (0, a) and (1, c). Similarly, since

$$r_2(0) = b\,(1-0) + d \times 0 = b$$
$$r_2(1) = b\,(1-1) + d \times 1 = d$$

it follows that the graph of $r_2(p)$ consists of the straight line segment joining (0, b) and (1,d). It will be seen that this makes the sketching of the graphs $r_1(p)$ and $r_2(p)$ a very easy matter. The graph of $E_R(p)$ is then also easily derived according to the following observation. The graph of $E_R(p)$ consists of that line which, for every permissible value of p, contains the lower of the two points $(p, r_1(p))$ and $(p, r_2(p))$.

EXAMPLE

For the Bombing sorties game

80%	100%
90%	50%

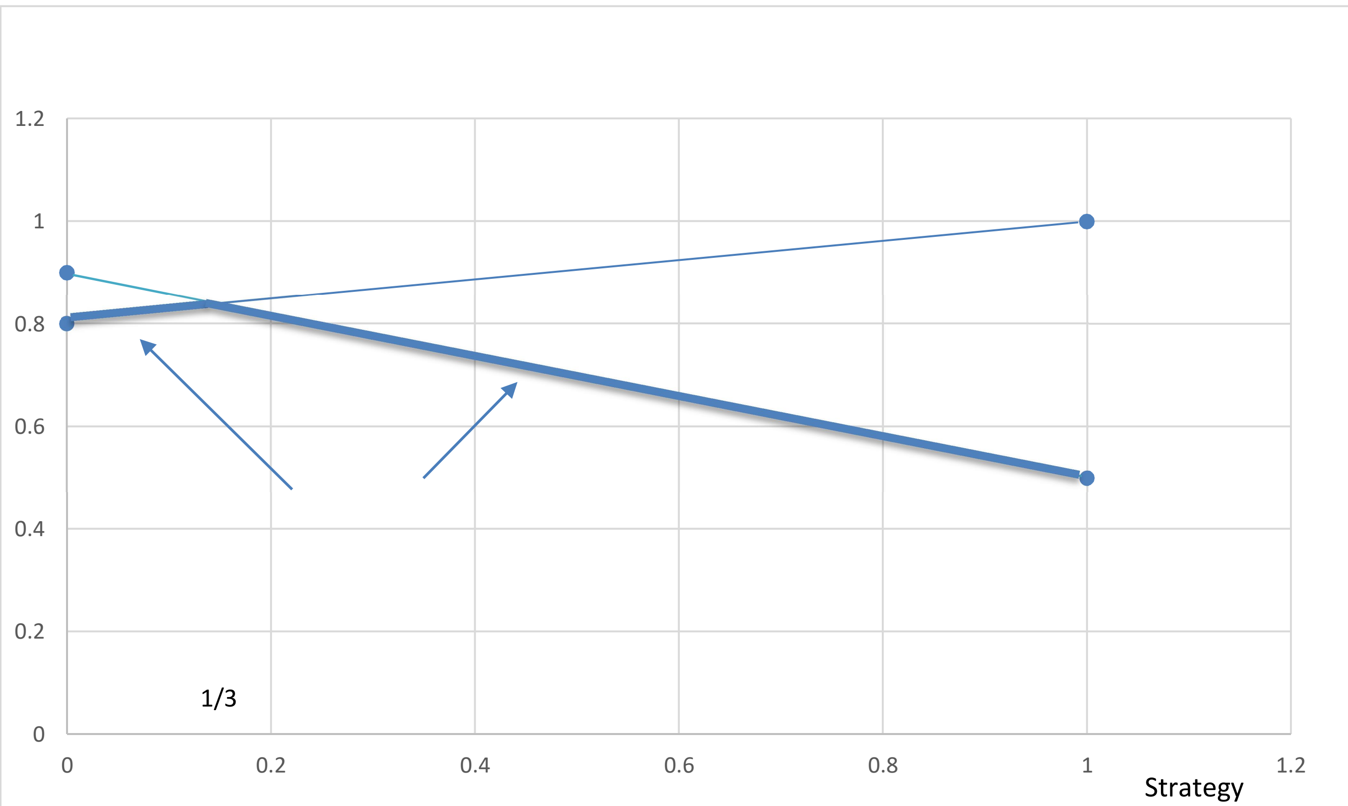
we have a =80% =.8, b =100% =1, c = 90% = .9, d =50% =.5,

$$r_1(p) = (.9-.8)\ p + .8 = .1p + .8,$$

$$r_2(p) = (.5-1)\ p + 1 = -.5p + 1$$

and so, the graph of $E_R(p)$ is the heavy broken line .

As the graph of $E_R(p)$ coincides with that of $r_1(p)$ for small values of p, it follows that when Ruth employs the strategy [1 — p, p] for small values of p,



is close to 1 Charlie should respond with [0,1]. The cut off point is of course the value of p that lies directly below the point of intersection of the graphs of $r_1(p)$ and r2(p) this important value is found by solving the equation,

$r_1(p)$ = r2(p)

or

$$.1p+8 = -.5p+1$$

$$.6p=.2$$

$$P = .2/.6 = 1/3$$

Hence,

[1,0] is an optimal counterstrategy for Charlie when $p \leq 1/3$.

[0,1] is an optimal counterstrategy for Charlie when $p \geq 1/3$.

In other words, as long as Ruth places bomb on the support plane less than 1/3 of the time. Charlie should persist in attacking the bomber. Once the bomb is placed on the support plane with a frequency greater than 1/3, Charlie should switch to attacking this weaker plane consistently. When p=1/3 all of Charlie's strategies will yield the same expected payoff.

Since the intersection of the graphs $r_1(p)$ and $r_2(p)$ also happens to be the highest points on the graph of $E_r(p)$, it also corresponds to Ruth's wisest choice. By employing the strategy,

$$[1-1/3,1/3] = [2/3,1/3]$$

Ruth obtains the largest possible expected payoff she can guarantee. The exact value of this largest expected payoff can be computed by substituting p=1/3 into either $r_1(p)$ and $r_2(p)$.

$r_1(1/3) = .1 \times 1/3 + .8 = .83333\ldots = 83.3\%$.

or

$r_2(1/3) = -.5 \times 1/3 + 1 = .83333\ldots = 83.3\%$.

It is clear from the foregoing examples that the highest point of the graphs $E_r(p)$ is of special strategic significance. unfortunately, this graph may have more than one highest point and so some care must be exercised in stating following theorem.

THEOREM

If (x, y) is any highest point on the graph of $E_r(p)$ then, [1-x, x] is a maximin strategy for Ruth, and y is Ruth's maximin expectation.

If Ruth employs the maximin strategy [1-x, x] then she can expect to win, on the average, at least y on each play.

b)THE MINIMAX STRATEGY

In game theory, minimax is a decision rule used to minimize the worst-case potential loss; in other words, a player considers all of the best opponent responses to his strategies, and selects the strategy such that the opponent's best strategy gives a payoff as large as possible.

Minimax is additionally valuable in combinatorial games, in which each position is allocated a result. The least complex model is doling out a "1" to a triumphant position and "-1" to a horrible one, however as this is hard to work out for everything except the easiest games, middle of the road assessments (explicitly picked for the game being referred to) are by and large essential. In this unique circumstance, the objective of the principal player is to augment the assessment of the position, and the objective of the subsequent player is to limit the assessment of the position, so the minimax rule applies.

We now turn to Charlie and devise for him too a good strategy. The graph of Charlie's expected payoff in 2×2 zero-sum games is obtained in much the same way as Ruth's graph, with some important differences, and similar conclusions can be drawn.

If one player of a game employs a fixed strategy, then the opponent has an optimal counterstrategy that is pure .For any strategy [1-q,q] that in the general 2×2 game.

a	b
c	d

Let c1(q) and c2(q) denote the expected payoffs that come from Ruth's pure strategies [1,0] and [0,1] respectively. The auxiliary diagrams are

	1-q	q
1	a	b
0	c	d

c1(q)

	1-q	q
0	a	b
1	c	d

c2(q)

the expected payoffs are;

$c1(q) = 1 \times (1-q) \times a + 1 \times q \times b = a(1-q) + bq = (b-a)q + a$

$c2(q) = 1 \times (1-q) \times c + 1 \times q \times d = c(1-q) + dq = (d-c)q + c$

Ec(q) is denotes by expected payoff selected from c1(q) and c2(q)

Ec(q) = the larger of { c1(q),c2(q) }

The independent variable ‘q’ appears in the expression of c1(q) and c2(q) with degree of at most 1,the graphs of these functions are straight lines. q denotes a probability then $0 \leq q \leq 1$ and these graphs consist of line segments that lie over the interval [0,1] on the q axis ie;

$c1(0) = a \times (1-0) + b \times 0 = a$

$c1(1) = a \times (1-1) + b \times 1 = b$

similarly,

$c2(0) = c \times (1-0) + d \times 0 = c$

$c2(1) = c \times (1-1) + d \times 1 = d$

The graph of c1(q) is the line segment joining the points (0,a) and (1,b) and the graph of c2(q) is the line segment joining the points (0,c) and (1,d)

The graph of Ec(q) consists of that line which, for every permissible value of 'q' contains the higher of the two points (q, c1(q)) and (q, c2(q))

EXAMPLE 1

For the Bombing-sorties game

80%	100%
90%	50%

We have ,

a = 80% = .8

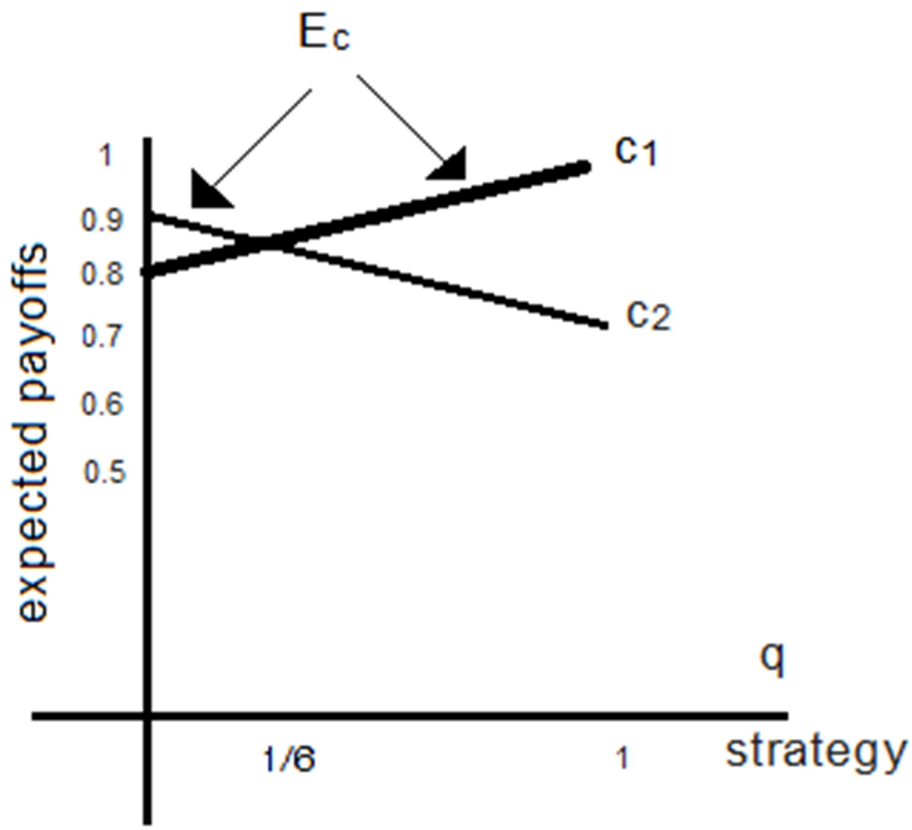
b = 100% = 1

c = 90% = .9

d = 50% = .5

c1(q) = (1-.8)q+.8 = .2q+.8

c2(q) = (.5-.9)q+.9= -.4q+.9



The graph of Ec(q) coincides with that of C2(q) for small values of q, it follows that when Charlie employs the strategy [1-q,q] for small values of q, Ruth should respond with the pure counterstrategy [0,1] ; when q is close to 1, Ruth should respond with [1,0].

The cutoff point is of course the value of q that lies directly below the point of intersection of the graphs of c1(q) and c2(q)

$$c1(q)= c2(q)$$

OR

$$.2q + .8 = -.4q + .9$$

$$.6q = .1$$

$$q = .1/.6 = 1/6$$

Hence ,

[0,1] is an counterstrategy for Ruth when $q \leq 1/6$

[1,0] is an counterstrategy for Ruth when $q \geq 1/6$

The intersection of the graphs of $c1(q)$ and $c2(q)$ also happens to be the lowest point on the graph of $Ec(q)$, it also corresponds to Charlie's wisest strategy by employing the strategy

$$[1- 1/6, 1/6] = [5/6, 1/6]$$

The exact value of this lowest expected payoff can be computed by substituting $q= 1/6$ into either $c1(q)$ or $c2(q)$:

$$C1(1/6) = .2\times 1/6 + .8 = .8333.... = 83.3\%$$

OR

$$C2\,(1/6) = -.4\times 1/6 + .9 = .8333.... = 83.3\%$$

Then we can conclude that the lowest point of the graph of $Ec(q)$ is of special significance.

THEOREM

If (x,y) is any lowest point on the graph of Ec(q), Then [1-x, x] is a minimax strategy for Charlie, and 'y' is Ruth's minimax expectation.

ie, If Charlie employs the minimax strategy [1-x,x] then he can expect to hold Ruth's average winning to no more that y

EXAMPLE

In the abstract game

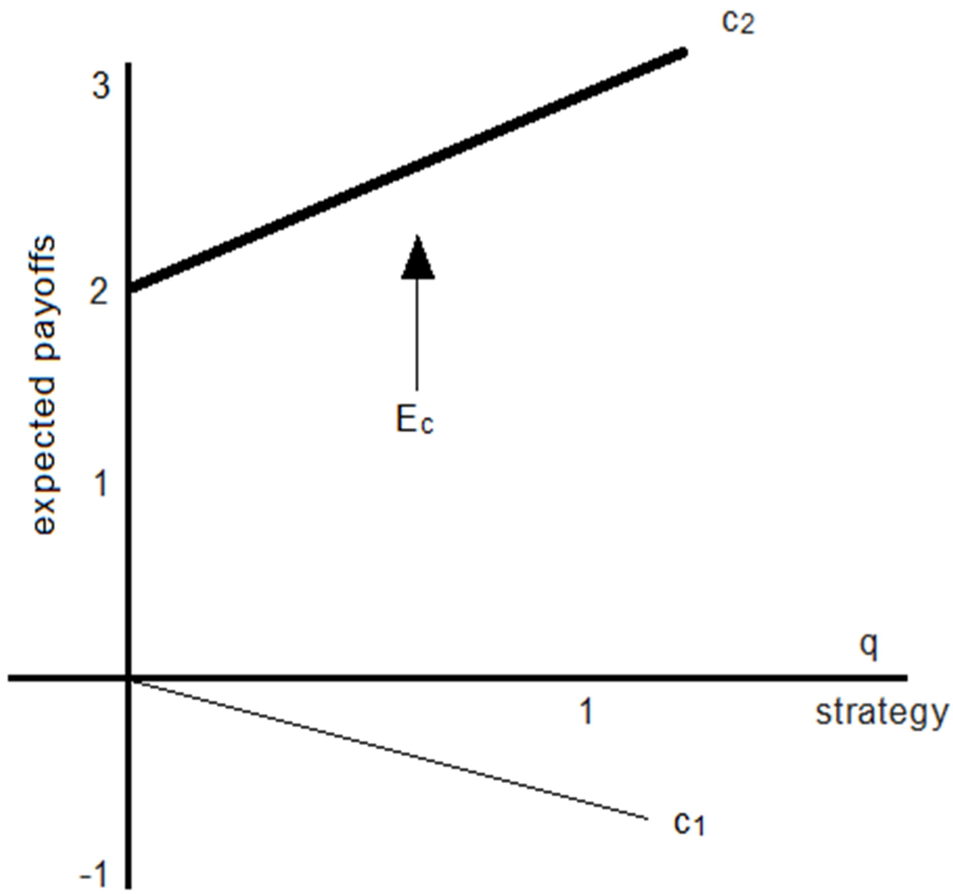
0	-1
2	3

a=0 b=-1 c=2 d=3

c1(q) = (-1-0)q+0 = -q

c2(q) = (3-2)q+2 = q+2

Then,



Here the graph of Ec(q) coincides with that of C2 (q). The low point on this graph corresponds to q= 0, the y- coordinate of (0,2) which is the lowest point of the graph of Ec (q) is 2.This minimax expectation constitutes a ceiling or an upper bound, on Ruth's expectation.

3. SOLUTIONS OF ZERO SUM GAMES

Zero-sum game is a mathematical representation in game theory and economic theory of a situation which involves two sides, where the result is an advantage for one side and an equivalent loss for the other. In other words, player one's gain is equivalent to player two's loss, therefore the net improvement in benefit of the game is zero.

THEOREM

For every 2 X 2 zero-sum game there is a single number v such that

i. The maximin strategy guarantees Ruth an expected payoff of at least v;
ii. The minimax strategy guarantees Charlie that Ruth's expected payoff will not exceed v.

When Ruth employs the maximin strategy and Charlie employs the minimax strategy, then Ruth can expect to win v units per play. For this reason, the number v is called the value of the game. The value of the game together with its maximin and minimax strategies constitutes the solution of the game. Thus , by example 2 of the previous two chapters is summarized by saying that the solution of Bombing-Sorties is :

Value = 83.3%

Minimum strategy = [2/3,1/3]

Minimax strategy = [5/6,1/6]

a) Strictly determined 2 X 2 games:

Those for which Ruth has pure maximin strategy (Charlie has a pure minimax strategy). Graphically, this means that the line segments corresponding to r1(p) and r2(p)either do not intersect at all, intersect at a common endpoint, or else their intersection point of is not the highest point on the graph of ER(P). The reason for this nomenclature is that the strategies being pure each player knows with complete certainty which option he will play next the one indicated by the pure strategy.

b) Nonstrictly determined 2 X 2 games:

All the other games, for which, necessarily, the maximin strategies (also the minimax strategies) are never pure. Graphically this means that the line segments corresponding to r1(p) and r2(p) intersect internally and the point of intersection is higher than any other point on the graph of ER(p). In other words, these line segments form an approximate figure X. Such is the case for both Penny-matching and Bombing-sorties.

THEOREM

A 2 X 2 zero-sum game is strictly determined if and only if it contains an entry s which is minimal for its row and maximal for its column.

An entry in a 2X2 game that is minimal for its row and maximal for its column is called a Saddle point. Thus ,the entry 2 is a saddle point of the game.

0	-1
2	3

THEOREM

The saddle point of a strictly determined 2X2 game is also the value, and its row and column constitutes pure maximin and minimax strategy.

The following procedures takes all the guess work and/or graphing out of the task of recognizing and solving strictly determined games. Given a game, write at the bottom of each column that column's maximum entry and write at the right of each row that row's minimum entry. if any of the column maxima equals any of the row minima, the game is strictly determined, that common entry is the saddle point and the value of the game, and its row and column constitute the respective pure maximin and minimax strategies of the game. This is the case for games a and c of table. Game b, however, is nonstrictly determined.

The 2 x 2 nonstrictly determined zero-sum game are subject to a solution procedure that is just as simple as the one that solves the strictly determined variety.

a	b
c	d

We define Ruth's oddments to be that one of the pairs

[d-c,a-b] or [c-d,b-a]

MINIMAX THEOREM

For every m x n zero-sum game there is a number v which has the following properties:

a. Ruth has a mixed strategy that guarantees her an expected payoff of at least v;
b. Charlie has a mixed strategy that guarantees that (Ruth's) expected payoff will be at most v.

The quantity v whose existence is asserted in theorem 6 is called the value of the game. The strategies mentioned in parts a and b of this theorem are called the *maximin* and the *minimax* strategies of the game. The value of the game together with its maximin and minimax strategies constitute its solution.

An entry in an m x n game that is minimal for its row and maximal for its column is called a saddlepoint. Thus the entries 2 and -1 are respective saddlepoints of the two games of table. The same bookkeeping method of comparing row minima with column maxima will locate these saddlepoints in the larger games just as well as it did for the 2 x 2 games.

A game with a saddlepoint is said to be strictly determined. The saddlepoint's entry is the value of the game and its row and column constitute Ruth's maximin and Charlie's minimax strategies.

LEMMA 1

Ruth has a pure maximin strategy if and only if G has a saddlepoint. In that case the payoff of the saddle point equals the maximum value.

PROOF

It may be assumed without loss of generality that a≤ b. suppose first that Ruth has a pure maximin strategy. The reader is reminded that the line segments $r_1(p) = (1-p)a + pc$ and $r_2(p) = (1-p)b + pd$, $0 \le p \le 1$, join the points (0,a) to (1,c) and (0,b) to (1,d) respectively. Since the maximin strategy that high point of $E_R(p)$ it follows from the assumption of its purity that either these line segments do not intersect in an interior point or they both have nonnegative slopes or they both have nonpositive slopes. Keeping in mind that $a \le b$ it is now easily verified that in these four cases the game G has saddle points at a,c,d,a respectively and that these payoff equal the maximin value of the game.

LEMMA 2

Charlie has a pure minimum strategy if and only if G has a raddle point. In that case the payoff of the saddle point equals the minimax value.

LEMMA 3

If the game G has no pure maximin strategies then it has

Maximin strategy = [d-c/a-b-c+d,a-b/a-b-c+d]

Maximin value = ad-bc/a-b-c+d

PROOF

Since G is assumed to not have a pure maximin strategy, it follows that the graphs of $r_1(p)$ and $r_2(p)$. as $r_1(p)=(1-p)a+pc$ and $r_2(p)=(1-p)b+pd$ this nonpure maximin strategy is found by solving the equation

$$(1-p)a+pc=(1-p)b + pd$$

Or

$$(a-b-c+d)p=a-b$$

For p. Since G is nonstrictly determined a-b-c+d≠0, and so p=(a-b)/(a-b-c+d) and 1-p =(d-c)/(a-b-c+d). the maximin value is then obtained by substituting this value of p into either $r_1(p)$ or $r_2(p)$.

LEMMA 4

If the game G has no pure minimax strategies, then it has
Minimax strategy=[d-b/a-b-c+d,a-c/a-b-c+d]
Minimax value= ad-bc/a-b-c+d
Now we can prove the Minimax theorem for 2 x 2 games

2 x 2 MINIMAX THEOREM

For every 2 x 2 zero-sum game there is a number v which has the following properties:

a. Ruth has a mixed strategy that guarantees her an expected payoff of at least v;
b. Charlie has a mixed strategy that guarantees that Ruth's expected payoff will be at most v.

PROOF

As noted above, if the game has a saddle point then this theorem follows from lemma 1 and lemma 2. If the game does not have a saddle point then it follows from lemma 1 and 2 that the game has neither a pure maximin nor a pure minimax strategy. Hence, by lemma 3 Ruth has a strategy that guarantees her an expected payoff of (ad-bc)/(a-b-c+d) and by lemma 4 Charlie has a strategy that guarantees that Ruth's expected payoff will not exceed (ad-bc)/(a-b-c+d). Thus, v= (ad-bc)/(a-b-c+d)

4. LARGER GAMES: 2 X n AND m X 2 GAMES

In this chapter, we are going to discuss on the solution of zero- sum games in which one of the players have only two options. We would end the chapter by giving an example based on the real life scenario put forward by the anthropologist W. C. A. Davenport.

a) 2 x n Games

When solving a 2 x n game, Charlie have n available pure strategies rather than just 2. Accordingly $r_j(p)$ denotes the expected payoff where j = 1, 2, . . . , where Ruth employs strategy [1 – p, p], $0 \leq p \leq 1$ and Charlie consistently selects the j- th column (pure strategy) and if $E_R(p)$ denotes the payoff Ruth can expect while employing [1 – p, p], then

$$E_R(p) = \text{the minimum of } \{r_1(p), r_2(p), \ldots, r_n(p)\}$$

and the graph of $E_r(p)$ coincides for each p with lowest of the points on the graph of $r_1(p), r_2(p), \ldots, r_n(p)$. If j- th column of the game is

g
h

then the graph of $r_j(p)$ consists of the line segment that joins the points (0, g) and (1, h).

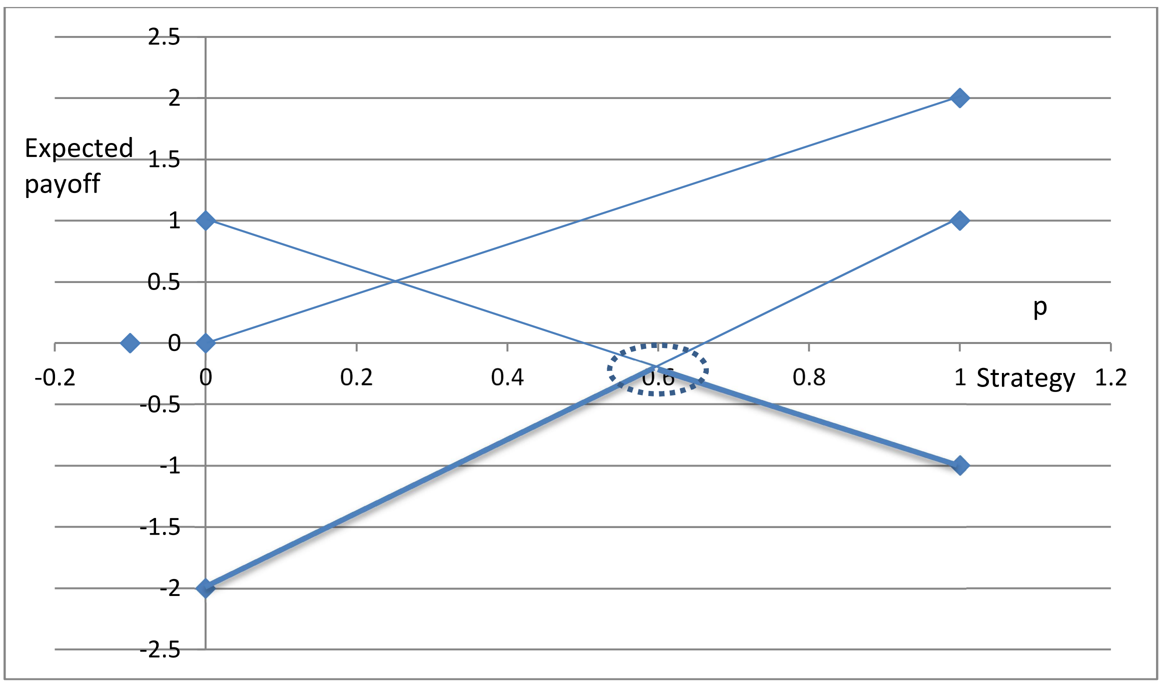
Theorem: - When solving a 2 x n zero- sum game, if a maximum strategy is determined by the point of intersection of two of the r_j's, then the corresponding value of p can be determined by the oddment method.

EXAMPLE

Determine the solution of the game

0	1	-2	*-2*
2	-1	1	*-1*
2	*-1*	*1*	

From the method of strictly determined games, it is seen that the game doesn't have a saddle point. Thus now we compute the graph



As the required point lies on r_2 and r_3, we can restrict our attention to the subgame

1	-2
-1	1

That consists of the second and third columns of the original game. This subgame is nonstrictly determined and has oddments [2, 3] for Ruth and[3, 2] for Charlie. The original game has maximin strategy [.4, .6], minimax strategy [0, .6, .4] and the auxiliary diagram

	0	*1*	*0*
.4	0	1	-2
.6	2	-1	1

yields the value

v = .4 x 1 x 1 + .6 x 1 x (-1) = .4 - .6 = -.2

b)m x 2 Games

Games of dimensions m x 2 are subject to a resolution that is similar to that of the 2 x n games, the main difference being that now it is Charlie's point of view that guides us. Accordingly, if for each i = 1, 2, . . . , m ci(q) denotes the expected payoff when Ruth consistently employs her i-th row against Charlie's arbitrary mixed strategy of [1-q, q], 0≤q≤1, and if Ec(q) denotes the expected payoff that Ruth will select under the circumstances.

Ec(q) = the maximum of {c1(q), c2(q), ... ,cm(q)}.

The graph of Ec(q) coincides for each q with the highest of the points on the graphs of c1(q), c2(q), ... ,cm(q). As was the case before if the i- th row of the game is

g	h

then the graph of Ec(q) consists of the line segment that joins (0,g) and (1,h).

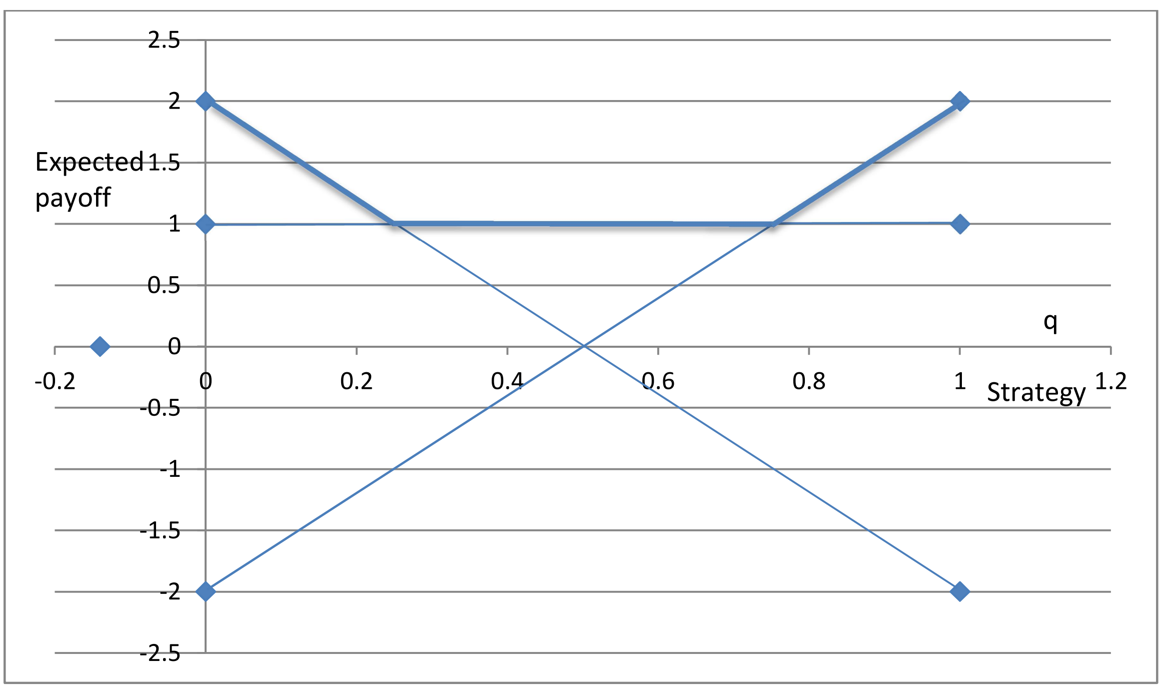
Theorem: - When solving a m x 2 zero- sum game, if a minimax strategy is determined by the point of intersection of two of the ci's, then the corresponding value of q can be determined by the oddments method.

EXAMPLE

Compute the solution of the game

1	1	*1*
-2	2	*-2*
2	-2	*-2*
2	*2*	

The graph of Ec(q) is given below



The left end point of the flat bottom comes from Charlie's oddments in the subgame determined by the first and third rows

1	1
-2	2

These oddments are [2 – 1, 1 – (-2)] = [1, 3] which yields a minimax strategy [1/4, 3/4], or q= ¾

The value of the game is found to be 1. Ruth can accomplish this by consistently choosing first row, i. e. with the pure strategy of [1, 0, 0].

c) The Jamaican Fishing Village.

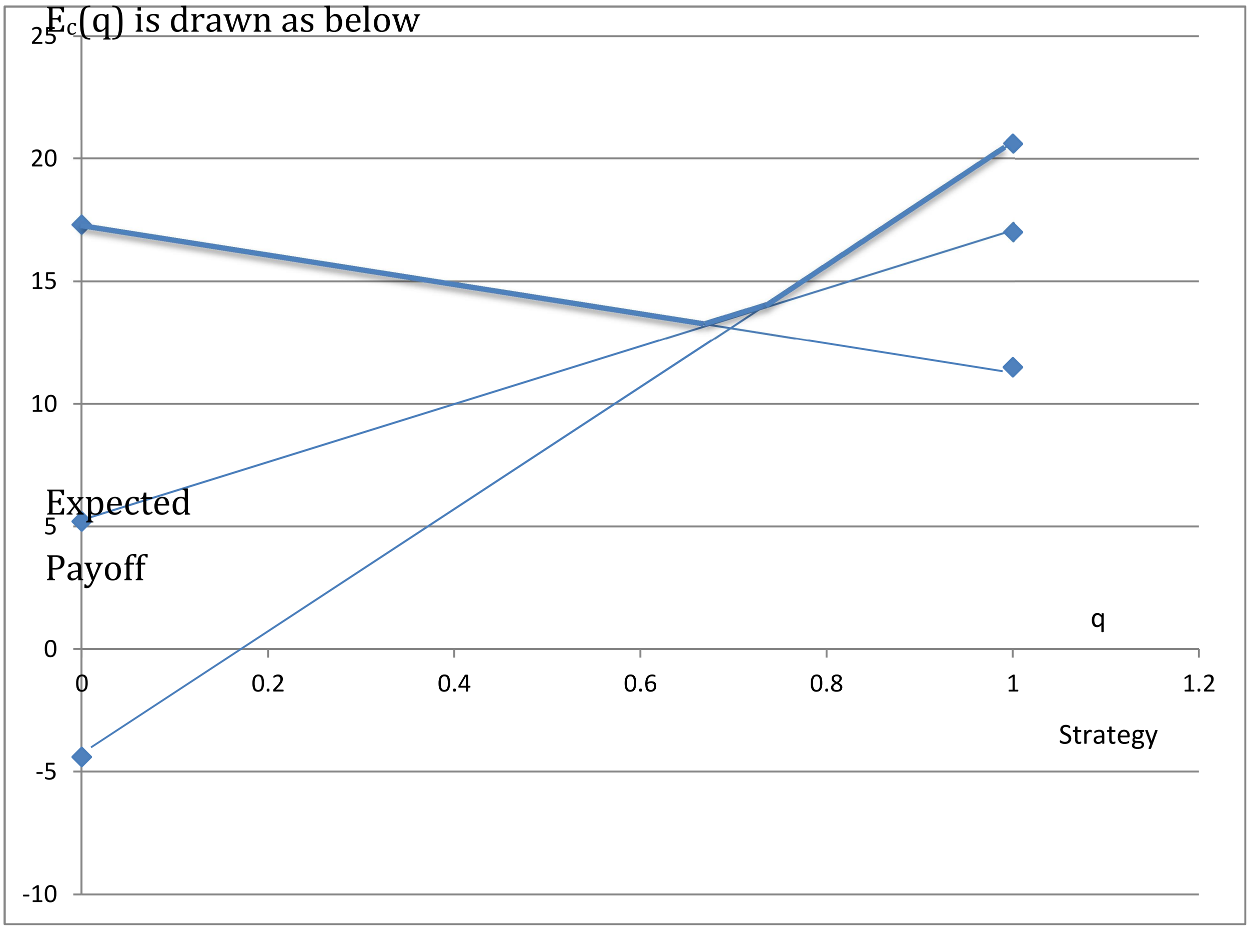
This example describes a game that was extracted by the anthropologist W. C. A. Davenport from his observation of the behaviour of the inhabitants of a certain Jamaican fishing village. These fishermen possessed twenty six fishing canoes manned each by a captain and two or three crewmen. The fishing grounds were divided into inside and outside banks. The inside bank lay from 5 to 15 miles offshore, whereas the outside banks lay beyond. The crucial factor that distinguished between the two areas was the occasional presence of very strong currents in the latter, which rendered fishing impossible. Accordingly, each captain must decide on a trap setting policy. He could

1. set all his pots inside,
2. set all the pots outside, or
3. set some of the pots inside and some outside.

Davenport modelled this situation as a 3 x 2 game whose players are the village and the environment. Each of the captains' selection on a fishing strategy along with the "decision" of the environment whether to send current or not, composes an estimated payoff in pounds.

		Environment	
		Current	No- current
	Inside	17.3	11.5
Village	In- out	5.2	17.0
	Outside	-4.4	20.6

Pretending that the environment is a conscious player, its expectation graph $E_c(q)$ is drawn as below



The requisite subgame consists of the first two rows of the given game

17.3	11.5
5.2	17.0

The village's oddments are [17.0 - 5.2, 17.3 – 11.5] = [11.8, 5.8] which in turn yields the maximum strategy of

$$[.67, .33, 0]$$

with a corresponding maximum expectation of

$$.67 \times 17.3 + .33 \times 5.2 = 13.31$$

Thus the Game Theory recommends the village to do 67% of its fishing inside the banks exclusively, 33% of it as a combination of inside- outside and none of the fishermen to dedicate themselves to fishing in the outside alone. This way the fishermen can be guaranteed an expected payoff of at least 13.31 pounds per outing.

Davenport observed that 18 (69%) of the captains fished only in the inside banks, 8 (31%) adopted the inside- outside combination, and none restricted their fishing to the outside bank alone. A remarkable fit between theory and observation.

CONCLUSION

Game theory, branch of applied mathematics that provides tools for analyzing situations in which parties, called players, make decisions that are interdependent. This interdependence causes each player to consider the other player's possible decisions, or strategies, in formulating strategy. A solution to a game describes the optimal decisions of the players, who may have similar, opposed, or mixed interests, and the outcomes that may result from these decisions.

Game theory has been applied to a wide variety of situations in which the choices of players interact to affect the outcome. In stressing the strategic aspects of decision making, or aspects controlled by the players rather than by pure chance, the theory both supplements and goes beyond the classical theory of probability.

As any principle of mathematics, game theory has its foundation with the consideration of 2 x 2 zero- sum games. We have learnt to find the floor value of a game using Maximin strategy and also to find the ceiling value of a game using Minimax strategy. Based on these fundamentals, we have constructed a generalization to finding the solution of zero- sum games in which one of the players has only two option whereas the other player can have more than two options.

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