A STUDY ON FUZZY DECISION MAKING

DISSERTATION SUBMITTED TO MAHATMA GANDHI UNIVERSITY,

KOTTAYAM

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF

BACHELOR OF SCIENCE IN MATHEMATICS

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2019-2022

DECLARATION

We hereby declare that this project entitled "**A STUDY ON FUZZY DECISION MAKING**" is a bonafide record of work done by us under the supervision of Ms. NATHASHA BABY, Guest Lecturer, Department of Mathematics, Bharata Mata College, Thrikkakara and the work has not previously formed by the basis for the award of any academic qualification, fellowship or other similar title of any other University or Board.

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<u>CERTIFICATE</u>

This is to certify that the project entitled **"A STUDY ON FUZZY DECISION MAKING"** submitted jointly by Alex Shaju, Jismi Jose, Melitta Johnson, Nikhil Paul, Siona Sabu and Vijishiya Vijayan in partial fulfillment of the requirements for the B.Sc. Degree in Mathematics is a bonafide record of the studies undertaken by them under my supervision at the Department of Mathematics, Bharata Mata College, Thrikkakara, during 2021 – 2022. This dissertation has not been submitted for any other degree elsewhere.

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ACKNOWLEDGEMENT

We have immense pleasure in presenting this dissertation on "**A STUDY ON FUZZY DECISION MAKING**". We would like to express our sincere gratitude towards Ms. NATHASHA BABY, Guest Lecturer in the Department of Mathematics of Bharata Mata College, Thrikkakara for her valuable support and guidance that enabled us to make study in this topic and to prepare this dissertation.

We are also thankful to Dr. Seethu Varghese, Head of the Department of Mathematics, for her valuable advices. We are obliged to all other teachers in the department for their cooperation to prepare this project.

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CHAPTER 1

INTRODUCTION

A fuzzy set is a class of objects with a continuum of grades of membership, such a set is characterized by a membership function which assigns to each object a grade of membership ranging between 0 and 1.The notions of inclusion, union, intersection, complements, relation, convexity, etc. are extended to such sets and various properties of these notions in the context of fuzzy sets are established. In particular, a separation theorem for convex fuzzy sets is proved without requiring that the fuzzy sets be disjoint.

Fuzzy sets allow us to represent vague concept in natural language. The fuzziness of a property lies in the lack of well-defined boundary of the set of objects to which this property applies.

More specifically consider a subset A, which is the set of all tall men in the universal set X that is the set of all men having the property of being tall. This fuzzy subset obviously has no well-defined boundary. Usually there are members of A, who are definitely tall, others who are not tall at all but there exist also border line cases. Then a membership of degree 1 is assigned to the objects that completely belong to A – here the men who are definitely tall. Conversely, the degree 0 is assigned to the objects that do not belong to A at all. Furthermore the membership degrees of borderline cases will naturally lie between 0 and 1. In other words the more element or object is characteristic A, its degree of membership is closer to 1. Therefore a fuzzy set is a set which has no sharp boundaries.

Making decisions is undoubtedly one of the most fundamental activities of human beings. We all are faced in our daily life with varieties of alternative actions available and we decide which of the available action to take. Initially, decision making has evolved into a respectable and rich field of study. The current literature on decision making, based largely on theories and methods developed in 18th century, is enormous.

The subject of decision making is the study of how decisions are actually made and how they can be made better or more successfully. The decision making process is of key importance for the functions such as inventory control, investment, personal actions etc.

Applications of fuzzy sets within the field of decision making consist of fuzzification of the classical theories of decision making. While decision making under conditions of risk have been modeled by probabilistic decision theories and game theories, fuzzy decision theories attempt to deal with the vagueness and non-specificity inherent in human formulation of preferences, constraints, and goals.

A decision is said to be made under conditions of certainty when the outcome for each action can be determined and ordered precisely. In this case, the alternative that leads to the outcome yielding the highest utility is chosen. The decision making problem becomes an optimization problem of maximizing the expected utility. When probabilities of the outcome are not known, or may not even be relevant, and outcomes for each action are characterized only approximately, we say that decisions are made under *uncertainty*. This is the prime domain for fuzzy decision making.

Several classes of decision making problems are usually recognized. According to one criterion, decision problems are classified as those involving a single decision maker and those which involve several decision makers. These problem classes are referred to as *individual decision making and multi-person decision making*, respectively.

PRELIMINARIES

- **Classical set /Crisp set (A)**

A set with fixed and well defined boundary.

Eg: A set of technical universities having at least five departments each.

- **Universe set / Universe of Discourse (X)**

A set consisting of all possible elements.

Eg: All technical universities in the world.

- **Characteristic function**

A set A subset of X defined by a function, usually called a Characteristic function, that declares with each element of x are members of the set and which are not. Set A is defined by its characteristic function μ_A; as follows:

$$\mu_A(x) = \begin{cases} 1 \ ; & if\ x \in A \\ 0 \ ; & if\ x \notin A \end{cases}$$

That is, $\mu_A: X \rightarrow \{0, 1\}$

- **Membership Function**

A characteristic function of crisp set assigns a value of either 1 or 0 each individual in the universal set, thereby discriminating between members and nonmembers of the crisp set under consideration.

The most commonly used range of values of membership functions is the unit interval [0, 1]. In this case ,each membership function maps elements of a given universal set X, which is always a crisp set, into real numbers in [0,1].

- **Fuzzy set**

The values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set. Larger values denote higher degrees of set membership, such a function is called a membership function and the set defined by it is a fuzzy set.

A fuzzy set kept in A is a function from universal set to [0, 1].

That is, $A: X \rightarrow [0, 1]$

For example: 1) Let X=N set of all natural numbers, define a fuzzy set A kept in N as, A (n) = 1/n $\forall\, n \in \mathbb{N}$

That is, A(1)=1 , A(2)=1/2 , A(3)=1/3 , etc.

2) Let X = R the set of all real numbers: Then a fuzzy set A kept in R is

$$A(\mu) = \begin{cases} 0 \ ; & if\ \mu \in Q \\ 1 \ ; & if\ \mu \notin Q \end{cases}$$

CHAPTER 2

INDIVIDUAL AND MULTIPERSON DECISION MAKING

I. INDIVIDUAL DECISION MAKING

Fuzziness can be introduced into the existing models of decision models in various ways. In the first paper on fuzzy decision making Bellman and Zadeh [1970] suggest a fuzzy model of decision making in which relevant goals and constraints are expressed in terms of fuzzy sets, and a decision is determined by an appropriate aggregation of these fuzzy sets. A decision situation in this model is characterized by the following components:

- a set A of *possible actions*;
- a set of *goals* G_i ($i \in \mathbb{N}_n$), each of which is expressed in terms of a fuzzy set defined on A;
- a set of *constraints* C_j ($j \in \mathbb{N}_m$) each of which is also expressed by a fuzzy set defined on A.

It is common that the fuzzy sets expressing goals and constraints in this formulation are not defined directly on the set of actions, but indirectly, through other sets that characterize relevant states of nature. Let G'_i and C'_j be fuzzy sets defined on sets X_i and Y_j respectively, where $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_m$. Assume that these fuzzy sets represent goals and constraints expressed by the decision maker. Then, for each $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_m$, we describe the meaning of actions in set A in terms of sets X_i and Y_j by functions

$$g_i : A \rightarrow X_i$$

$$c_j : A \rightarrow Y_j$$

and express goals G_i and constraints C_j by the compositions of gi with G'_i and the composition of c_j and C'_j; that is,

$$G_i (a) = G'_i (g_i (a)), \tag{2.1}$$

$$C_j (a) = C'_j (c_j (a)) \tag{2.2}$$

for each $a \in A$.

Given a decision situation characterized by fuzzy sets A, G_i ($i \in \mathbb{N}_n$), and Cj ($j \in \mathbb{N}_m$), a *fuzzy decision*, D, is conceived as a fuzzy set on A that simultaneously satisfies the given goals

$$D(a) = \min \left[\inf_{i \in \mathbb{N}n} Gi(a), \inf_{j \in \mathbb{N}m} Cj(a) \right] \tag{2.3}$$

F or all $a \in A$, provided that the standard operator of fuzzy intersection is employed.

Once a fuzzy decision has been arrived at, it may be necessary to choose the "best" single crisp alternative from this fuzzy set. This may be accomplished in a straightforward manner by choosing an alternative $\hat{a} \in A$ that attains the maximum membership grade in D. Since this method ignores information concerning any of the other alternatives, it may not be desirable in all situations. When A is defined on $\mathbb{R}$, it is preferable to determine $\hat{a}$ by an appropriate defuzzification method.

Example:

Suppose that an individual need to decide which of four possible jobs, a_1, a_2, a_3, a_4, to choose. His or her goal is to choose a job that offers a high salary under the constraints that the job is interesting and within close driving distance. In this case $A = \{a_1, a_2, a_3, a_4\}$, and the fuzzy sets involved represent the concepts of *high salary under, interesting job, and closed driving distance*. These concepts are highly subjective and context-dependent, and must be defined by the individual in a given context. The goal is expressed in monetary terms, independent of the jobs available. Hence, according to our notation, we denote the fuzzy set expressing the goal by G' is given in Fig. 2.1, where we assume, for convenience, that the underlying universal set is $\mathbb{R}^+$. To express the goal in terms of set A, we need a function $g: A \rightarrow \mathbb{R}^+$, which assigns to each job the respective salary. Assume the following assignments:

$g(a_1) = \$40,000$,

$g(a_2) = \$45,000$,

$g(a_3) = \$50,000$,

$g(a_4) = \$60,000$.

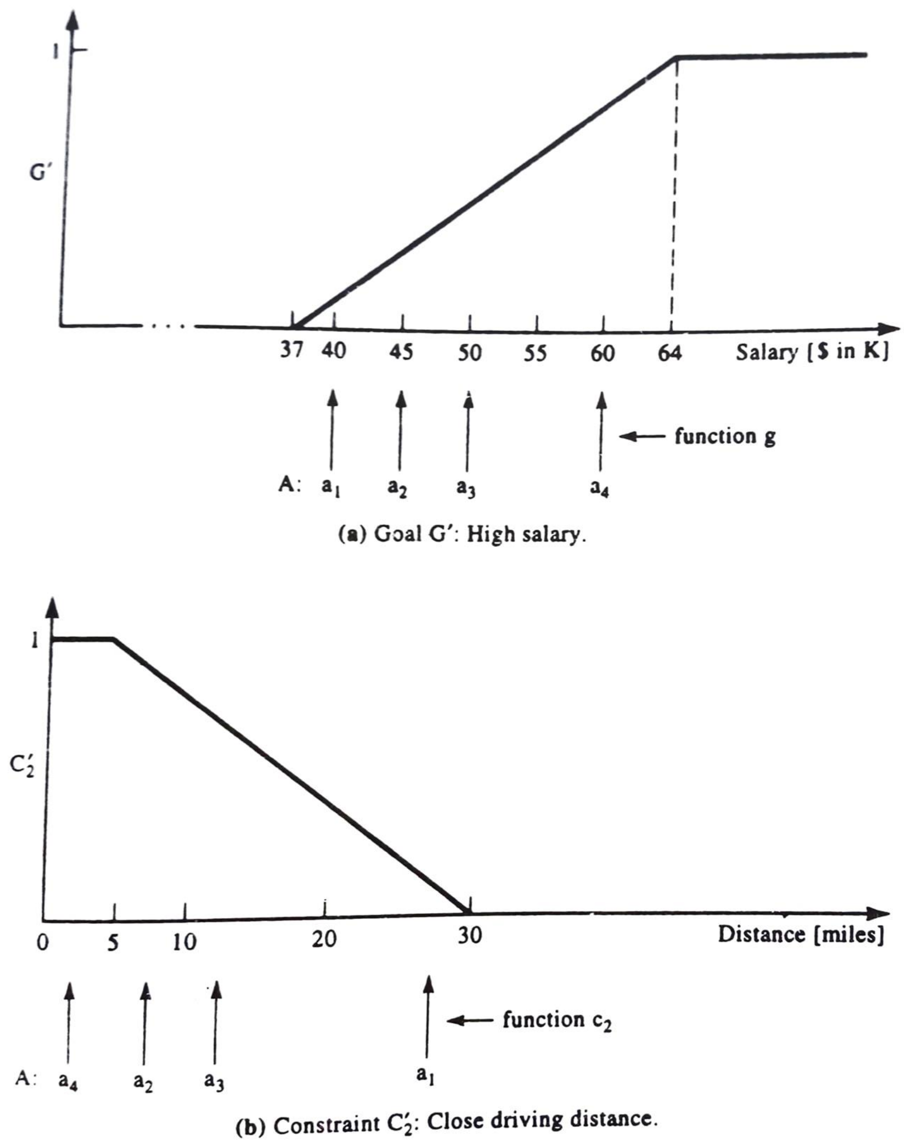


Fig 2.1 Fuzzy goal and constraint: (a) goal G': high salary; (b) constraint C'_2: close driving distance

Composing now function g and G', we obtain the fuzzy set

$$G = .11/a_1 + .3/a_2 + .48/a_3 + .8/a_4$$

which expresses the goal in terms of the available jobs in set A.

The first constraint, requiring that the job be interesting, is expressed directly in terms of set A. Assume that the individual assigns to the four jobs in A the following membership grades in the fuzzy set of interesting jobs:

$$C_1 = .4/a_1 + .6/a_2 + .2/a_3 + .2/a_4.$$

The second constraint, requiring that the driving distance be close, is expressed in terms of the driving distance from home to work. Following our notation, we denote the fuzzy set expressing this constraint by C'_2. A possible definition of C'_2 is given in Fig. 2.1, where distances of the four jobs are also shown. Specifically,

$$c_2\,(a_1) = 27 \text{ miles},$$

$$c_2\,(a_2) = 7.5 \text{ miles},$$

$$c_2\,(a_3) = 12 \text{ miles},$$

$$c_2\,(a_4) = 2.5 \text{ miles}.$$

By composing functions c2 and C'_2, we obtain the fuzzy set

$$C_2 = .1/a_1 + .9/a_2 + .7/a_3 + 1/a_4,$$

which expresses the constraint in terms of the set A.

Applying now formula (2.3), we obtain the fuzzy set

$$D = .1/a_1 + .3/a_2 + .2/a_3 + .2/a_4.$$

which represents a fuzzy characterization of the concept of *desirable job*. The job to be chosen is $\hat{a} = a_2$; this is the most desirable job among the four available jobs under the given goal G and constraints C_1, C_2 as expressed by (2.3).

II. MULTI PERSON DECISION MAKING

When decision made by more than one person are modeled, two differences from the case of single decision maker can be considered: first, the goal of the individual decision makers may differ such that each places a different ordering on the alternatives; second, the individual decision makers may have access to different information upon which to base their decision. Theories known as *n*-person game theories deal with both of these considerations, team theories of decision making deal only with the second, and group decision theories deal only with the first.

A fuzzy model group decision was proposed by Blin [1974] and Blin and Whinston [1973]. Here each member of a group of *n* individual decision makers is assumed to have a reflexive, antisymmetric and transitive preference ordering $P_{k,}$ $k \in \mathbb{N}_n$, which totally or partially orders a set X of alternatives. A "social choice" function must then be found which, given the individual preference orderings, produces the most acceptable overall group preference ordering. Basically, this model allows for the individual decision makers to possess different aims and values while still assuming that the overall purpose is to reach a common, acceptable decision. In order to deal with the multiplicity of opinion evidenced in the group, the social preference S may be defined as a fuzzy binary relation with membership grade function

$$S: X \times X \rightarrow [0, 1],$$

which assigns the membership grade S ($x_{i,}$ x_j), indicating the degree of group preference of alternative x_i over $x_{j.}$ The expression of this group preference requires some appropriate means of aggregating the individual preferences. One simple method computes the relative popularity of alternatives xi over x_j by dividing the number of persons preferring x_i to x_j, denoted by N ($x_{i,}$ x_j), by the total number of decision makers, *n*. This scheme corresponds to the simple majority vote. Thus,

$$S\ (xi,\, x_j) = \frac{N\ (Xi.\ Xj)}{n}. \qquad (2.4)$$

Other method of aggregating the individual preference may be to accommodate different degree of influence exercised by the individuals in the group. For instance, a dictatorial situation can be modeled by the group preference relation S for which

$$S\ (x_i, x_j) = \begin{cases} 1\ if\ xi \overset{k}{>} xj\ \ for\ some\ individual\ k \\ \quad 0\ \ otherwise \end{cases}$$

where $\overset{k}{>}$ represents the preference ordering of the one individual *k* who exercises complete control over the group decision.

Once the fuzzy relationship S has been defined, the final non-fuzzy group preference can be determined by converting S into its resolution form

$$S = \bigcup_{\alpha \in [0,1]} \alpha\, {}^{\alpha}S$$

which is the union of the crisp relations ${}^{\alpha}S$ comprising of the α-cuts of the fuzzy relation S, each scaled by α. Each value α essentially represented the level of agreement between the individuals concerning the particular crisp ordering ${}^{\alpha}S$. One procedure that maximize the final agreement level consist of intersecting the classes of crisp total orderings that are compatible with the pairs in the α-cuts ${}^{\alpha}S$ for increasingly smaller value of α until a single crisp total ordering is achieved. In this process, any pair $\langle x_i, x_j \rangle$ that leads to an intransitivity are removed. The largest value α for which the unique compatible ordering on X × Y is found represents the maximized agreement level of the group and the crisp ordering itself represents the group decision.

Pairwise comparisons

In this method, $f(x_i, x_j)$, denotes the attractiveness grade given by the individual to x_i with respect to x_j. These primitive evaluations, which are expressed by positive numbers in a given range, are made by the individual for all pairs of alternatives in the given set X. They are then converted to relative preference grades, $F(x_i, x_j)$, by the formula.

$$F(x_i, x_j) = \frac{f(xi,\ xj)}{\max[f(xi,\ xj),\ f(xj,\ xi)}$$

$$= \min[1,\ f(xi,\ xj) / f(xj,\ xi)]$$

for each pair $(x_i, x_j) \in X^2$. Clearly, $F(x_i, x_j) \in [0, 1]$ for all pairs $(x_i, x_j) \in X^2$.

When $F(x_i, x_j) = 1$, x_i is considered at least as attractive as x_j. Function F, which may be viewed as a membership function of a fuzzy relation on X, has for each pair $(x_i, x_j) \in X^2$ the property

$$\max[F(x_i, x_j). F(x_j, x_i)] = 1.$$

The property means: for each pair of alternatives, at least one must be as attractive as the other.

For each $x_i \in X$, we can now calculate the overall relative preference grades, $p(x_i)$, of x_i with respect to all other alternatives in X by the formula

$$p(x_i) = \min_{xj \in X} F(x_i, x_j),$$

The preference ordering of alternatives in X is then induced by the numerical ordering of these grades $p(x_i)$.

Example 1

Consider a group of people involved in a business partnership who intend to buy a common car for business purposes. To decide what car to buy is a multi-person decision problem. Assume, for the sake of simplicity that only five car models are considered. Acclaim, Accord, Camry, Cutlass and Sable. Assume further that, using the numbers suggested in table (a) for specifying the attractiveness grades the evaluation prepared by one person in the group is given in table (b) the corresponding relative preference grades and the overall relatives preference grade are given in(c).The latter induce the following preference ordering of the models: Camry, Sable, Accord, Cutlass and Acclaim. Orderings expressing preferences by the other members of the group can be determined in the similar way. Then the method for multi person decision making describe in the section can be applied to these preference orderings to obtain a group decision.

(a) Suggested numbers for attractiveness grading.

f(xi,xj)	Attractiveness of xi with respect to xj
1	Little attractive
3	Modertely attractive
5	Strongly attractive
7	Very strongly attractive
9	Extremely attractive
2,4,6,8	Intermediate value between levels

(b) Given attractiveness grades

f(xi,xj)	Acclaim	Accord	Camry	Cutlass	Sable
Acclaim	1	7	9	3	8
Accord	3	1	3	2	4
Camry	1	1	1	3	5
Cutlass	2	7	7	1	7
Sable	2	6	8	3	1

(c) Relative preference grade and overall relative preference grades

F(xi,xj)	Acclaim	Accord	Camry	Cutlass	Sable	p(xi)
Acclaim	1	0.43	0.11	0.67	0.25	0.11
Accord	1	1	0.33	1	1	0.33
Camry	1	1	1	1	1	1
Cutlass	1	0.29	0.43	1	0.43	0.29
Sable	1	0.66	0.625	1	1	0.63

In this example $F(x_i, x_j)$ denotes the attractiveness grades given by the individual to x with respect to xj. These primitive evaluations which are expressed by the positive numbers in a given range are made by the individual for all pair of alternatives in the given set X they are the converted to relative preference grades $F(x_i, x_j)$ by the formula

$$F\ (x_i, x_j) = \frac{f\ (xi,\ \ xj)}{\max\ [\ f\ (xi,\ \ xj),\ \ f\ (xj,\ \ xi)}$$

$$= \min\ [1,\ \ f\ (xi,\ \ xj)\ /\ \ f\ (xj,\ \ xi)\ ]$$

for each pair (x_i, x_j)

For each $x_i \in X$, we can now calculate the overall relative preference grades, p (x_i), of x_i with respect to all other alternatives in X by the formula

$P\ (x_i) = \min F\ (x_i, x_j)$

The preference ordering of alternatives in X is then induced by the numerical ordering of these grades p (x_i).

CHAPTER 3

MULTICRITERIA AND MULTISTAGE DECISION MAKING

I. MULTICRITERIA DECISION MAKING

In multicriteria decision problems, relevant alternatives are evaluated according to a number of criteria. Each criterion induces a particular ordering of the alternatives, and we need a procedure by which to construct one overall preference ordering .There is a visible similarity between these decision problems and problems of multiperson decision making. In both cases, multiple orderings of relevant alternatives are involved and have to be integrated into one global preference ordering .The difference is that the multiple orderings represent either preferences of different people or ratings based on different criteria .The number of criteria in multicriteria decision making is virtually always assumed to be finite.

Let $X = \{x_1, x_2,..., x_n\}$ and $C = \{c_1, c_2,\ldots, c_m\}$ be, a set of alternatives and a set of criteria characterizing a decision situation, respectively. Then the basic information involved in multicriteria decision making can be expressed by the matrix.

$$R = \begin{bmatrix} r11 & r12 & \ldots & r1n \\ r21 & r22 & \ldots & r2n \\ \ldots\ldots & & & \\ rm1 & rm2 & \ldots & rmn \end{bmatrix}$$

Assume first that all entries of this matrix are real numbers in [0, 1] and each entry r_{ij} expresses the degree to which criterion c_i is satisfied by alternative x_j ($i \in \mathbb{N}_m$, $j \in \mathbb{N}_n$). Then R may be viewed as a matrix representation of a fuzzy relation on C × X.

It may happen that, instead of matrix R with entries in [0, 1], an alternative matrix $R' = [r'_{ij}]$, whose entries are arbitrary real numbers. R' can be converted to the desired matrix R by the formula.

$$r_{ij} = \frac{r'ij - \min_{j \in \mathbb{N}n} r'ij}{\max_{j \in \mathbb{N}n} r'ij - \min_{j \in \mathbb{N}n} r'ij}$$

for all $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$.

The most common approach to multicriteria decision problems is to convert them to single-criterion decision problems. This is done by finding a global criterion, $r_j = h(r_{1j}, r_{2j}, \ldots, r_{mj})$, that for each $x_j \in X$ is an adequate aggregate of values $r_{1j}, r_{2j}, \ldots, r_{mj}$ to which the individual criteria $c_1, c_2, \ldots, c_m$ are satisfied.

Example

Problem of recruiting and selecting personal

In this particular problem, the selection of conditions from a given set of individuals say $x_1, x_2, \ldots, x_n$ is guided by comparing candidates profiles with a required profile in terms of given criteria $c_1, c_2, \ldots, c_m$. This results in matrix R.

The entries r_{ij} of R expressed for each $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$, the degree to which candidate x_j conforms to the required profile in terms of criterion c_j. A frequently employed operator is the weighted average.

$$r_j = \frac{\sum_{i=1}^{m} \tilde{w}_i \tilde{r}_{ij}}{\sum_{i=1}^{m} \tilde{w}_i} \quad (j \in \mathbb{N}_n)$$

Where $w_1, w_2, \ldots, w_m$ are weights that indicates the relative importance of criteria $c_1, c_2, \ldots, c_m$. A class of possible weighted aggregations is given by the formula

$$r_j = h(r_{1j}^{w1}, r_{2j}^{w2}, \ldots, r_{mj}^{wm}),$$

where h is an aggregation operator and $w_1, w_2, \ldots, w_m$ are weights.

A more general situation in which the entries of matrix R are fuzzy number $\tilde{r}_{ij}$ on $\mathbb{R}^+$, and weights are specified in terms of fuzzy numbers $\tilde{w}_i$ on [0, 1]. Then, using the operations of fuzzy addition and fuzzy multiplication, we can calculate the weighted average $\tilde{r}_j$ by the formula

$$\tilde{r}_j = \sum_{i=1}^{m} \tilde{w}_i \tilde{r}_{ij}$$

Since fuzzy numbers are not linearly ordered, a ranking method is needed to order the resulting fuzzy numbers $\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n$.

II. MULTISTAGE DECISION MAKING

Multistage decision making is a sort of dynamic process. A required goal is not achieved by solving a single problem, but by solving a sequence of decision – making problems. A decision problem conceived in terms of fuzzy dynamic programing is viewed as a decision problem regarding a fizzy finite state automaton. One restrictions of the automaton in dynamic programing is that the state – transition relation is crisp and, hence characterized by the usual state-transition function of classical automata. The automaton operates with fuzzy input state and fuzzy internal state, and it is thus fuzzy in this sense. Another restriction is that no special output is needed. That is the next internal state is also utilized as output and; consequently the two need not be distinguished.

The automaton, $\mathcal{A}$, involved in fuzzy dynamic programing is defined by the triple

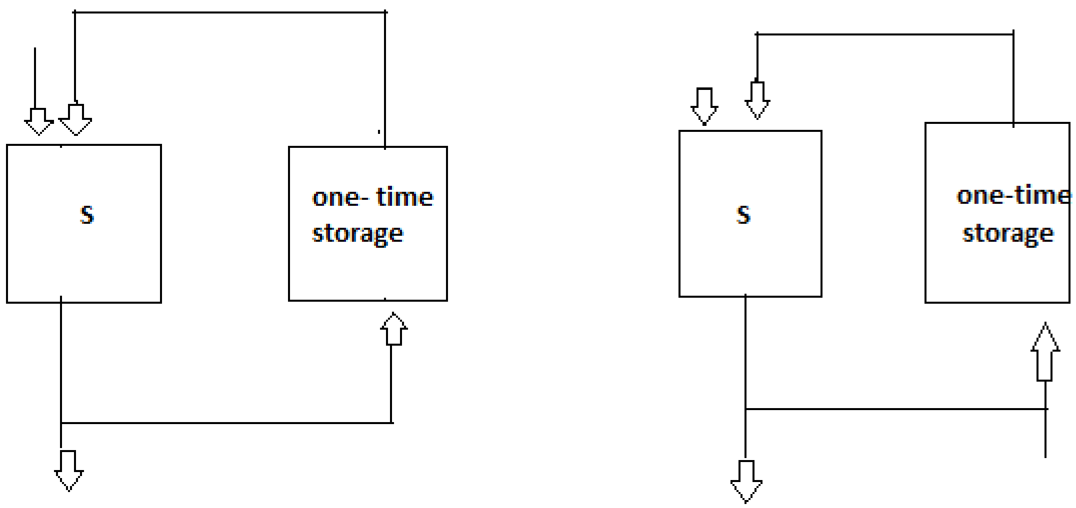
$$\mathcal{A} = \langle X, Z, f \rangle$$

where X and Z are respectively, the sets of input states and output states of $\mathcal{A}$, and

$$f: Z \times X \rightarrow Z$$

is the state-transition of A, whose meaning is to define, for each discrete time t ($t \in \mathbb{N}$), the next internal state, z^{t+1}, of the automaton in terms of its present internal state, z^t, and its present input state , x^t. That is,

$$z^{t+1} = f\,(z^t, x^t).$$



(a) Crisp automaton Fig.3.1 (b) Fuzzified automaton

A scheme of the described automaton is shown in Fig. 3.1. This type of automata is used in classical dynamic programming. For fuzzy dynamic programming they must be fuzzified by using the extension principle. A scheme of the fuzzified version is shown in Fig 3.1b, where A^t, C^t denote , respectively, the fuzzy input state and fuzzy internal state at time t, and C^{t+1} denote, the fuzzy internal state at time $t+1$. Clearly, A^t is a fuzzy set on X, while C^t and C^{t+1} are fuzzy sets on Z.

In this conception of decision making, the desired goal is expressed in terms of a fuzzy set C^N, where N is the time of termination of the decision process. The value of N, which defines the number of stages in the decision process, is assumed to be given. It is also assumed that the input of $\mathcal{A}$ is expressed at each time t by a fuzzy state A^t and that a particular crisp initial internal state z^0 is given

Considering fuzzy input states $A^0, A^1, \ldots, A^{N-1}$ as constraints and fuzzy internal state C^N as fuzzy goal in a fuzzy decision making, we may conceive of a fuzzy decision as a fuzzy set on X^N defined by

$$D=\tilde{A}^0 \cap \tilde{A}^1 \cap \ldots\ldots \tilde{A}^{N-1} \cap \tilde{C}^N,$$

where $\tilde{A}^t$ is a cylindric extension of A^t from X to X^N for each t = 0, 1,..., N-1, and $\tilde{C}^N$ is the fuzzy set on X^N that induces C^N on Z. That is, for any sequence $x^0, x^1, \ldots, x^{N-1}$, viewed as a sequence of decisions, the membership grade of D is defined by

$$D\,(x^0, x^1,..., x^{N-1}) = \min\,[A^0(x^0), A^1(x^1), ..., A^{N-1}(x^{N-1}), C^N(z^N)],$$

where z^N is uniquely determined by $x^0, x^1,....,x^{N-1}$, this definition assumes , of course ,that we use the standard operator of intersection. The decision problem is to find a sequence $x^0, x^1,...., x^{N-1}$ of input states such that

$$D\,(\hat{x}^0, \hat{x}^1,...., \hat{x}^{N-1}) = \max_{x0,....,xN-1} D(x^0, x^1,......, x^{N-1}).$$

To solve this problem by fuzzy dynamic programming, we need to apply a principle known in dynamic programming as the *principle of optimality* which can be expressed as follows: An optimal decision sequence has the property that whatever the initial state and initial decision are, the remaining decision must constitute an optimal policy with regard to the state resulting from the first decision.

Applying the principle of optimality and substituting for D, we can write

$$D\,(\hat{x}^0, \hat{x}^1,...., \hat{x}^{N-1}) = \max_{x0,....,xN-2} \{\max_{xN-1} \min\,[A^0(x^0), A^1(x^1),...., A^{N-1}(x^{N-1}), C^N(f(z^{N-1},x^{N-1}))]\}$$

This equation can be rewritten as

$$D\,(\hat{x}^0, \hat{x}^1,..., \hat{x}^{N-1}) = \max_{x0,....,xN-2} \{\min\,[A^0(x^0), A^1(x^1),..., A^{N-2}(x^{N-1}), \max_{xN-1} \min\,[A^{N-1}(x^{N-1}), C^N(f(z^{N-1},x^{N-1}))]]\}$$

$$= \max_{x0,....,xN-2} \{\min\,[A^0(x^0), A^1(x^1),..., A^{N-2}(x^{N-2}), \max_{xN-1} \min\,[A^{N-1}(x^{N-1}), C^N(z^N)]]\}$$

$$= \max_{x0,....,xN-2} \{\min\,[A^0(x^0), A^1(x^1),..., A^{N-2}(x^{N-2}), C^{N-1}(z^{N-1})]\}$$

$$\text{where } C^{N-1}(z^{N-1}) = \max_{xN-1} \min[A^{N-1}(x^{N-1}), C^N(z^N)].$$

Repeating this *backward iteration*, we obtained the set of N recurrence equations

$$C^{N-k}(z^{N-K}) = \max_{xN-k} \min[A^{N-k}(x^{N-k}), C^{N-k+1}(z^{N-k+1})],$$

for k=1, 2,...., N, where

$$z^{N-k+1} = f(z^{N-k}, x^{N+1}).$$

Hence, the optimal sequence $\hat{x}^0$, $\hat{x}^1$,..., $\hat{x}^{N-1}$ of decisions can be obtained by successively maximizing values $x^{N-k,}$ for k = 1, 2,..., N. This results successively in values $\hat{x}^{N-1}$,..., $\hat{x}^1$, $\hat{x}^0$.

Example:

Let us consider automaton with X = {x_1, x_2}, Z = {z_1, z_2, z_3}, and the state transition function expressed by the matrix

$$\begin{bmatrix} z1 & z2 \\ z3 & z1 \\ z1 & z3 \end{bmatrix}$$

whose entries are next internal states or any given present internal and output states. Assume that

N = 2, and the fuzzy goal at t = 2 is

$$C^2 = .3/z_2 + 1/z_2 + .8/z_3.$$

Assume further that the fuzzy constraints at input at times t = 0 and t = 1 are

$$A° = .7/x_1 + 1/x_2$$

$$A^1 = 1/x_1 + .6/x_2$$

To solve this decision problem, we need to find a sequence $\hat{x}^0$, $\hat{x}^1$ of input states for which the maximum,

$$\max_{\hat{x}0,\hat{x}1} \min[A^0(x^0), A^1(x^1), C^2(f(z^1, z^1))],$$

is obtained. Applying the first backward iteration for t = 1, we obtain

$C^1 (z_1) = \max \{\min [A^1 (x_1), C^2 (f (z_1, x_1))], \min [A^1 (x_2), C^2 (f (z_1, x_2))]\}$

$= \max \{\min [A^1 (x_1), C^2 (z_2)], \min [A^1 (x_2), C^2 (z_2)]\}$

$= \max \{\min [1, .3], \min [.6, 1)\}$

$= .6$

$C^1 (z_2) = \max \{\min [A^1 (x), C^2 (f (z_2, x_1))], \min [A' (x_2), C^2 (f (z_2, x_2))]\}$

$= \max \{\min [A'(x_1), C^2 (23)], \min [A' (x_2), C^2 (z_1)]\}$

$= \max \{\min [1, .8], \min [.6, .3]\}$

$= .8$

$C^1 (z_3) = \max \{\min [A^1 (x_1), C^2 (f (z_3, x_1))], \min [A' (x_2), C^2 (f (z_3, x_2))]\}$

$= \max \{\min [A^1 (x_1), C^2 (z_1)], \min [A^1 (x_2), C^2 (z_3)]\}$

$= \max \{\min [1, .3], \min [.6, .8]\}$

$= .6$

Hence,

$$C^1 = .6/z_1 + .8/z_2 + .6/z_3.$$

By maximizing the expression

$$\min [A^1(x^1), C^2 (f (z^1,x^1))],$$

we find the following best decision $\hat{x}^1$ for each state $z^1 \in Z$ at time $t = 1$:

z^1	z_1	z_2	z_3
$\hat{x}^1$	x_1	x_2	x_3

Applying now the second backward iteration for $t = 0$, we obtain

$C^0 (z_1) = \max \{\min [A^0 (x_1), C^1 (f (z_2, x_1))], \min [A^0 (x_2), C^1 (f (z_1, x_2))]\}$

$= \max \{\min [A^0 (x_1), C^1 (z_1)], \min [A^0 (x_2), C^1 (z_2)]\}$

$= \max \{\min [.7, .6], \min [1, .8]\}$

$= .8$

$C^0 (z_2) = \max \{\min [A^0 (x_1), C^1 (f (z_2, x_1))], \min [A^0 (x_2), C^1 (f (z_2, x_2))]\}$

$= \max \{\min [A^0 (x_1), C^1 (z_3)], \min [A^0 (x_2), C^1 (z_1)]\}$

$= \max \{\min [.7, .6], \min [1, .6]\}$

$= .6$

$C^0 (z_3) = \max \{\min [A^0 (x_1), C^1 (f (z_3, x_1))], \min [A^0 (x_2), C^1 (f (z_3, x_2))]\}$

$= \max \{\min [A^0 (x_1), C^1 (z_1)], \min [A^0 (x_2), C^1 (z_3)]\}$

$= \max \{\min [.7, .6], \min [1, .6]\}$

$= .6$

Hence,

$$C^0 = .8/z_1 + .86/z_2 + .6/z_3.$$

By maximizing the expression

$$\min [A^0 (x_0), C^1 (f (z^0, x^0))],$$

we find the following best decision $\hat{x}^0$ for each state $z^0 \in Z$ at time $t = 0$;

z^0	z_1	z_2	z_3
$\hat{x}^0$	x_2	x_1 or x_2	x_1 or x_2

The maximizing decisions for different initial states z^0 are summarized. For example, when the initial state is z_1, the maximizing decision is to apply action x_2 followed by x_1. In this case, the goal is satisfied to the degree

$C^0 (z_1) = \min [A^0 (x_2), C^1 (z_2)]$

$= \min [A^0 (x_2), \min [A^1 (x_1), C^2 (z_3)]]$

$= \min [A^0 (x_2), A^1 (x_1), C^2 (z_3)]$

$= \min [1, 1, .8]$

$= .8$

That is, the degree to which the goal is expressed in terms of $C^0 (z_1)$, where z_1 is the initial state. When the initial state is z_2, we have two maximizing decision; hence, there are two ways calculating $C^0 (z_2)$:

$C^0 (z_2) = \min [A^0 (x_1), A^1 (x_2), C^2 (z_3)]$

$= \min [.7, .6, .8]$

$= .6$

$C^0 (z_2) = \min [A^0 (x_2), A^1 (x_2), C^2 (z_2)]$

$= \min [1, .6, 1]$

$= .6$

That is, this goal is satisfied to the degree .6 when the initial states is z_2, regardless of which of to the maximizing decisions is used.

CHAPTER 4

FUZZY LINEAR PROGRAMMING

The classical linear programming problem is to find the minimum or maximum values of a linear function under constraints represented by linear inequalities or equations. The most typical linear programming problem is:

Minimize (or maximize) $c_1x_1 + c_2x_2 + ... + c_nx_n$

Subject to $a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \leq b_2$

…………………………………

$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \leq b_m$

$x_1, x_2, \ldots, x_n \geq 0$

The function to be minimized (or maximized) is called an objective function; let us denote it by z. The numbers c_i ($i \in \mathbb{N}_n$.) are called cost coefficient, and the vector $c = \langle c_1, c_2, ..., c_n \rangle$ is called a cost vector. The matrix $A = [a_{ij}]$, where $i \in \mathbb{N}_m$, and $j \in \mathbb{N}_n$, is called a constraint matrix, and the vector $v = \langle b_1, b_2, ..., b_n \rangle$ is called a right-hand side vector. Using this notation, the formulation of the problem can be simplified as

$$\text{Min } z = cx$$

$$\text{s.t } Ax \leq b$$

$$x \geq 0, \qquad (4.1)$$

where $x = \langle x_1, x_2, ..., x_n \rangle^T$ is a vector of variables and s.t. stands for "subject to." The set of vectors x that satisfy all given constraints is called a feasible set.

<u>Example:</u>

Let us consider,

$$\text{Min } z = x_1 - 2x_2$$

$$\text{s.t. } 3x_1 - x_2 \geq 1$$
$$2x_1 + x_2 \leq 6$$
$$0 \leq x_2 \leq 2$$
$$0 \leq x_1$$

Using fig.4.1 as a guide, we can show graphically how the solution of this linear programming problem can be obtained. First, we need to determine the feasible set. Employing an obvious geometrical interpretation, the feasible set is obtained in fig. 4.1 by drawing straight lines representing the equations $x_1 = 0$, $x_2= 0$, $x_2= 2$, $3x_1 - x_2 = 1$, and $2x_1 + x_2= 6$. These straight lines, each of which constraints the whole plane into a half-plane, express the five inequalities in our examples. When we take the intersection of the five- half planes, we obtain the shaded area in fig 4.1, which represents the feasible set. This area is always a convex polygon.

To find the minimum of the objective function z within the feasible set, we can draw a family of parallel lines representing the equation $x_1- 2x_2 = p$, where p is a parameter, and observe the direction in which p decreases. Then, we can imagine a straight line parallel to the others moving in that direction until it touches either an edge or a vertex of the convex polygon. At that point, the value of parameter p is the minimum value of the objective function z. If the requirements were to maximize the objective function, we would move the line in the opposite direction, the direction in which p increases.

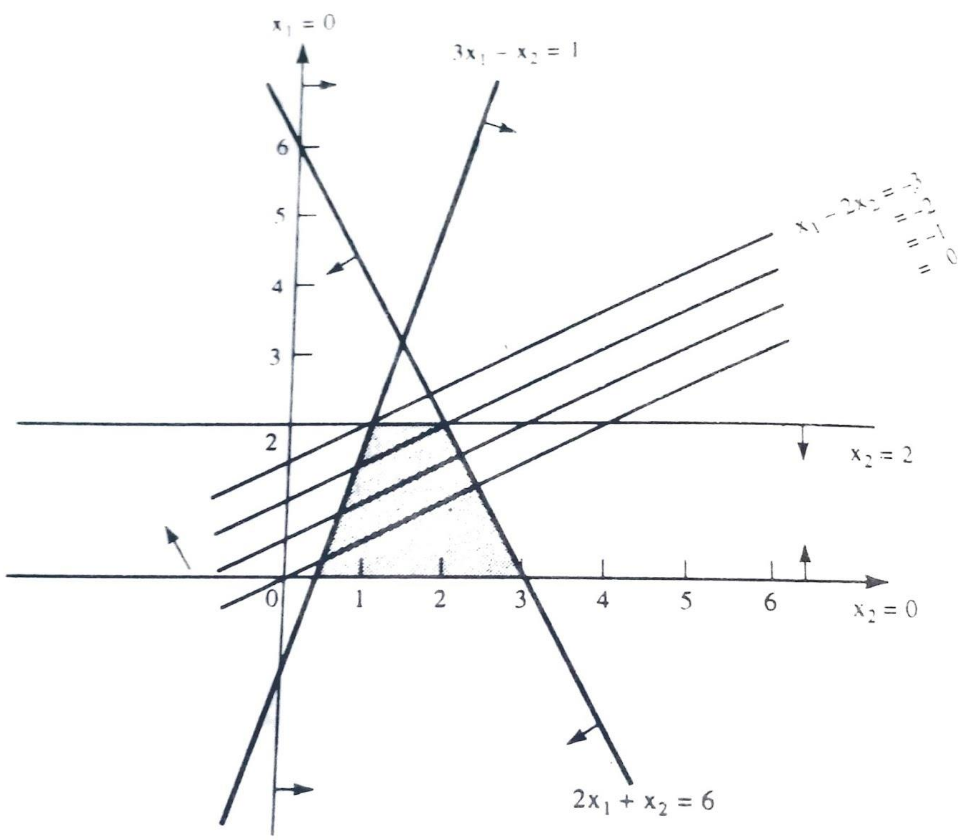


Fig: 4.1

In many practical situations, it is not reasonable to require that the constraints or the objective function in linear programming problems be specified in precise, crisp terms. In such situations, it is desirable to use some type of fuzzy linear programming. The most general type of fuzzy linear programming is formulated as follows:

$$\max \sum_{j=1}^{n} C_j X_j$$

$$\text{s.t.} \sum_{j=1}^{n} A_{ij} X_j \leq B_i \quad (i \in \mathbb{N}_m) \tag{4.2}$$

$$X_j \geq 0 \; (j \in \mathbb{N}_n)$$

where A_{ij}, B_i, C_j, fuzzy numbers, and X_j, are variables whose states are fuzzy numbers ($i \in \mathbb{N}_m$, $j \in \mathbb{N}_n$); the operations of addition and multiplication are operations of fuzzy arithmetic and $\leq$ denotes the ordering of fuzzy numbers.

Case 1. Fuzzy linear programming problems in which only the right-hand-side numbers B_i are fuzzy numbers:

$$\text{Max} \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \leq B_i \; (i \in \mathbb{N}_m) \tag{4.3}$$

$$x_j \geq 0 \; (j \in \mathbb{N}_n)$$

Case 2. Fuzzy linear programming problems in which the right-hand-side numbers B_i and the coefficients A_{ij} of the constraint matrix are fuzzy numbers:

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \sum_{j=1}^{n} A_{ij} X_j \leq B_i \quad (i \in \mathbb{N}_m) \tag{4.4}$$

$$x_j \geq 0 \quad (j \in \mathbb{N}_n)$$

In general, fuzzy linear programming problems are first converted into equivalent crisp linear or nonlinear problems, which are then solved by standard methods. The final results of a fuzzy linear programming problem are thus real numbers, which represent a compromise in terms of the fuzzy numbers involved.

Let us now discuss fuzzy linear programming problems of type (4.3). In this case, fuzzy numbers B_i ($i \in \mathbb{N}_m$) typically have the form

$$B_i\,(x) = \begin{cases} 1\,; & \text{when } x \le bi \\ \frac{bi+pi-x}{pi}\,; & \text{when } bi < x < b + pi \\ 0\,; & \text{when } b + P \le x, \end{cases}$$

where x ∈ ℝ. For each vector $x = \langle x_1, x_2, \ldots, x_n \rangle$, we first calculate the degree, $D_i(x)$, to which x satisfies the i^{th} constraint ($i \in \mathbb{N}_m$) by the formula

$$D_i(x) = B_i\,\left(\sum_{j=1}^{n} a_{ij}x_j\right)$$

These degrees are fuzzy sets on $\mathbb{R}^n$, and their intersection $\bigcap_{i=1}^{m} D_i$, is a fuzzy feasible set.

Next, we determine the fuzzy set of optimal values. This is done by calculating the lower and upper bounds of the optimal values first. The lower bound of the optimal values, z_l, is obtained by solving the standard linear programming problem:

$$\max z = cx$$

$$\text{s.t.} \sum_{j=1}^{n} a_{ij}x_j \le b_i \quad (i \in \mathbb{N}_m)$$

$$x_j \ge 0 \quad (j \in \mathbb{N}_n)\,;$$

the upper bound of the optimal values, z_u, is obtained by a similar linear programming problem in which each b_i is replaced with $b_i + p_i$:

$$\max z = cx$$

$$\text{s.t.} \sum_{j=1}^{n} a_{ij}x_j \le b_i + p_i \quad (i \in \mathbb{N}_m)$$

$$x_j \ge 0 \quad (j \in \mathbb{N}_n)\,.$$

Then, the fuzzy set of optimal values, G, which is a fuzzy subset of ℝ", is defined by,

$$G(x) = \begin{cases} 1\,; & \text{when } zu \le cx \\ \frac{cx - zl}{zu - zl}\,; & \text{when } zl \le cx \le zu \\ 0\,; & \text{when } cx \le zl. \end{cases}$$

Now the problem (4.3) becomes the following classical optimization problem:

$$\max \boldsymbol{\lambda}$$

$$\text{s.t. } \boldsymbol{\lambda}\,(z_u - z_l) - cx \le -z_l$$

$$\boldsymbol{\lambda} p_i + \sum_{j=1}^{n} a_{ij}x_j \le b_i + p_i \quad (i \in \mathbb{N}_m)$$

$$\boldsymbol{\lambda},\ x_j \ge 0 \ (j \in \mathbb{N}_n).$$

The above problem is actually a problem of finding x ∈ R" such that

$$[(\bigcap_{i=1}^{m} D_i) \cap G]\,(x)$$

reaches the maximum value; that is, a problem of finding a point which satisfies the constraints and goal with the maximum degree.

CONCLUSION

Fuzziness can be found in many areas of daily life such as in engineering, in medicine, in manufacturing and others. It is frequent, however in all areas in which human judgment evolution and decision are important. These are the areas of decision making, reasoning, learning etc.

In this project our aim is to study the applicability of fuzzy set theory to main classes of decision making problems. Several classes of decision making problems are there. According to one criterion, decision problems are classified as those involving single decision maker and which involve several decision makers.

Fuzzy goals and fuzzy constraints can be defined precisely as fuzzy sets in the space of alternatives. A fuzzy decision, then may be viewed as an intersection of the given goals and constraints. A maximizing decision is defined as a point in the space of alternatives at which the membership function of a fuzzy decision attains its maximum value.

The use of these concepts is illustrated by examples involving multistage decision processes in which the system under control is either deterministic or stochastic. By using dynamic programming, the determination of a maximizing decision is reduced to the solution of a system of functional equations. A reverse-flow technique is described for the solution of a functional equation arising in connection with a decision process in which the termination time is defined implicitly by the condition that the process stops when the system under control enters a specified set of states in its state space.

Almost all of these enable us to know how decisions are made and how they made better or more successfully.

REFERENCES

1. George J. Klir and Bo-Yuan, Fuzzy Sets and Fuzzy Logic - Theory and Applications, Prentice-Hall, Inc., 2010.

2. George J. Klir and Tina A. Folger, Fuzzy sets, Uncertainty, and Information, Prentice-Hall, Inc., 1987.

3. H.J. Zimmermann, Fuzzy Set Theory and its Applications, Alled Publishers Limited, 1991