"A STUDY ON INNER PRODUCT SPACE"

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DECLARATION

We here by declare that this project entitled A STUDY ON INNER PRODUCT space is a bonafide record of work done by us under the supervision of Miss Riya Aliyas and the work has not previously formed the basis for any other qualification, fellowship, or other similar title of other university or board.

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This is to certify that the dissertation entitled A STUDY ON INNER PRODUCT SPACES submitted jointly by Mr.Nitto Raphel , Mr. Phijo Poulose, Miss.Anna Sandra Varghese , Miss.Dhanya M U, Miss. Karthika V, Miss. Sethulakshmi Bhasi in partial fulfillment of the requirements for the Bsc Degree in MATHEMATICS is a bonafide record of the studies undertaken by them under my supervision at the Department of Mathematics, Bharata Mata College, Thrikkakara during 2019-2022. This dissertation has not been submitted for any other degree elsewhere.

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CONTENTS

ABSTRACT

The goal for this project is to allow us to understand the various aspects of Inner product spaces. It will give a deep insight on analysis, especially functional analysis and it has many applications in branches including matrix algebra, algebraic geometry etc. It is an essential tool of functional analysis and vector theory, allow analysis of classes of functions rather than individual functions.

INTRODUCTION

INNER PRODUCT SPACE

In mathematics, an **inner product space** (or, rarely, a Hausdorff pre-**Hilbert space**) is a real vector space or a complex vector space with an operation called an inner product. The inner product of two vectors in the space is a scalar, often denoted with angle brackets such as in $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$. Inner products allow formal definitions of intuitive geometric notions, such as lengths, angles, and orthogonality (zero inner product) of vectors. Inner product spaces generalize Euclidean vector spaces, in which the inner product is the dot product or scalar product of Cartesian coordinates. Inner product spaces of infinite dimension are widely used in functional analysis. Inner product spaces over the field of complex numbers are sometimes referred to as **unitary spaces**. The first usage of the concept of a vector space with an inner product is due to Giuseppe Peano, in 1898.

Peano called his vector spaces "linear systems" because he correctly saw that one can obtain any vector in the space from a linear combination of finitely many vectors and scalars — av + bw + … + cz. A set of vectors that can generate every vector in the space through such linear combinations is known as a spanning set. The dimension of a vector space is the number of vectors in the smallest spanning set.

The linearity of vector spaces has made these abstract objects important in diverse areas such as statistics, physics, and economics, where the vectors may indicate probabilities, forces, or investment strategies and where the vector space includes all allowable states.

An inner product naturally induces an associated norm, (denoted **|x|** and **|y|**); so, every inner product space is a normed vector space. If this normed space is also complete (that is, a Banach space) then the inner product space is a Hilbert space. If an inner product space H is not a Hilbert space, it can be extended by completion to a Hilbert space $\overline{\boldsymbol{H}}$ .This means that **H** is a linear subspace of $\overline{\boldsymbol{H}}$, the inner product of **H** is the restriction of that of $\overline{\boldsymbol{H}}$ and **H** is dense in $\overline{\boldsymbol{H}}$ for the topology defined by the norm.

PRELIMINARIES

Basic definitions

Vector space

A vector space is a set whose elements, often called vectors, may be added together and multiplied by numbers called scalars. Scalars are often real numbers, but can be complex numbers or, more generally, elements of any field.

Subspace

A Subspace is a Vector Space included in another larger Vector Space. Therefore, all properties of a Vector Space, such as being closed under addition and scalar multiplication still hold true when applied to the Subspace.

Norm

A norm is a function from a real or complex vector space to the non-negative real numbers that behaves in certain ways like the distance from the origin

Closed set

Closed set can be defined as a set which contains all its limit points.

Convex set

A subset of a Euclidean space is convex if, given any two points in the subset, the subset contains the whole line segment that joins them. Equivalently, a convex set or a convex region is a subset that intersects every line into a single line segment.

Linearly independent

A set of vectors is said to be linearly dependent if there is a nontrivial linear combination of the vectors that equals the zero vector .If no such linear combination exists, then the vectors are said to be linearly independent.

Span

The span of a set of vectors is the set of all linear combinations of the vectors.

Eigen spaces

A set of the eigenvectors associated with a particular eigen value, together with the zero vector.

Chapter 1

INNER PRODUCT SPACE

An *inner product space* (or *pre-Hilbert space*) is a vector space **X** with an inner product defined on **X**. A Hilbert space is a complete inner product space. Here, an **inner product** on **X** is a mapping of X×X into the scalar field K of X; that is, with every pair of vectors x and y there is associated a scalar which is written

$$\langle x, y \rangle$$

and is called the inner product' of x and y, such that for all vectors x, y, z and scalars α we have

(IP1) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(IP2) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(IP3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(IP4) $\langle x, x \rangle \geq 0$

$\langle x, x \rangle = 0 \quad \Leftrightarrow \quad x = 0$

An inner product on X defines a norm on X given by

(1) $||x|| = \sqrt{\langle x, x \rangle}$ (≥ 0)

and a metric on X given by

(2) $d(x, y) = || x - y || = \sqrt{\langle x - y, x - y \rangle}$.

Hence inner product spaces are normed spaces.

In (IP3), the bar denotes complex conjugation. Consequently, if X is a real vector space, we simply have

$$\langle x, y\rangle = \langle y, x\rangle$$

Theorem 1

If x ϵ H and y ϵ **H**, where **H** is an inner product space, then

(1) $|(x, y)| \leq ||x||\,||y||$

and

(2) $||\, x + y \,|| \leq ||x|| + ||\, y||$.

Moreover

(3) $||y|| \leq ||\, \boldsymbol{\lambda} x + y \,||$ for every $\boldsymbol{\lambda} \in ⊄$

if and only if $x \perp y$.

Theorem 2

Every nonempty closed convex set **E** ⊂ **H** contains a unique x of minimal norm.

PROOF

The parallelogram law

(1) $\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$ ($x \in H, y \in H$)

follows directly from the definition $\| x \|^2 = (x, x)$. Put

(2) $d = \inf \{ \| x \| : x \in E \}$.

Choose $x_n \in E$ so that $\|x_n\| \to d$. Since $\frac{1}{2}(x_n + x_m) \in E$, $\|x_n + x_m\|^2 > 4d^2$. If x and y are replaced by x_n and x_m in (1), the right side of (1) tends to $4d^2$. Hence (1) implies that $\{x_n\}$ is a Cauchy sequence in *H*, which therefore converges to some $x \in E$, with $\| x \| = d$.

If y = E and $\| y \| = d$, the sequence (x, y, x, y,...) must converge, as we just saw. Hence y = x.

Theorem 3

If M is a closed subspace of H, then

$H = M \oplus M^{\perp}$.

The conclusion is, more explicitly, that *M* and $M^{\perp}$ are closed subspaces of *H* whose intersection is {0} and whose sum is *H*. The space $M^{\perp}$ is called the *orthogonal complement* of *M*.

PROOF

If $E \subset H$, the linearity of (x, y) as a function of x shows that $E^{\perp}$ is a subspace of *H*, and the Schwarz inequality (1) of Theorem 1 implies then that $E^{\perp}$ is closed.

If $x \in M$ and $x \in M^{\perp}$, then (x, x) = 0; hence x = 0. Thus $M \cap M^{\perp} = \{0\}$.

If $x \in H$, apply Theorem 2 to the set x - *M* to conclude that there exists $x_1 \in M$ that minimizes $\| x - x_1 \|$.Put $x_2 = x - x_1$. Then $\| x_2 \| \leq \| x_2 + y \|$ for all $y \in M$. Hence $x_2 \in M^{\perp}$, by Theorem 1 Since $x = x_1 + x_2$, we have shown that $M + M^{\perp} = H$.

Corollary

If M is a closed subspace of *H*, then

$(M^{\perp})^{\perp}$ = *M*.

PROOF

The inclusion $M \subset (M^{\perp})^{\perp}$ is obvious. Since

$M \oplus M^{\perp}$ = H = $M^{\perp} \oplus (M^{\perp})^{\perp}$,

M cannot be a proper subspace of $(M^{\perp})^{\perp}$.

We now describe the dual space *H** of *H*.

Chapter 2

ORTHOGONAL AND ORTHONORMAL SETS

In a metric space ***X***, the *distance* δ from an element $x \epsilon$ ***X*** to a nonempty subset ***M*** $\subset$ ***X*** is defined to be

$\delta = \text{Inf}_{\tilde{y} \in M} \; d(x, \tilde{y})$ ($M \neq \emptyset$)

In a normed space this becomes

(1)$\delta = \text{Inf}_{\tilde{y} \in M} \; \| x - \tilde{y} \|$($M \neq \emptyset$)

(2) $$\delta = \| x - y \|,$$

Direct sum

A vector space *X* is said to be *direct sum* of two subspaces ***Y*** and ***Z*** o f *X*, written

X = ***Y***⊕***Z***

If each $x \in X$ has a unique representation

$$x = y + z$$

$y \epsilon$ ***Y,*** $z \epsilon$ ***Z.***

Then ***Z*** is called an *algebraic compliment* of ***Y*** in *X* and vice versa, and ***Y***, ***Z*** is called a *complementary pair* of subspaces in *X.*

Orthonormal Sets

An *orthogonal* set ***M*** in an inner product space is a subset ***M*** ⊂***X*** whose elements are pair wise orthogonal. An *orthonormal* set ***M*** ⊂***X*** is an orthogonal set ***X*** in whose elements have norm 1, that is, for all *x, y* ∈*M,*

(1) $\langle x, y\rangle = \begin{cases} 0, if x \neq y \\ 1, \; if \; x = y \end{cases}$

If an orthogonal or orthonormal set ***M*** is countable, we can arrange it in a sequence (x_n) and call it an *orthogonal* or *orthonormal sequence*, respectively.

More generally, an indexed set, or *family*, (x_α), α ∈I, is called *orthogonal* if $x_\alpha \perp x_\beta$ for all α, β ∈ Iwe have

(2)$\langle x_\alpha, x_\beta\rangle = \delta_{\alpha\beta} = \begin{cases} 0, \; if \; x \neq y \\ 1, \; if \; x = y \end{cases}$

Here, $\delta_{\alpha\beta}$ is the Kronecker delta.

For orthogonal elements x, y we have $\langle x, y\rangle = 0,$ so that we readily obtain the Pythagorean relation

(3)$\|x + y\| = \|x\|^2 + \|y\|^2$

Linear independence

An orthonormal set is linearly independent.

PROOF

Let $\{ e_1, . . . , e_n\}$ be orthonormal and consider the equation

$$\alpha_1 e_1 + . . . + \alpha_n e_n = 0$$

Multiplication by a fixed $e_j$gives

$$\left\langle \sum_k a_k e_k, e_j \right\rangle = \sum_k a_k \langle e_k, e_j \rangle = \alpha_j \langle e_j, e_j \rangle = \alpha_j = 0$$

And proves linear independence for any finite orthonormal set. This also implies linear independence if the given *orthonormal* set is infinite, by the definition of linear independence.

Examples

Euclidean space R^3. In the space R^3 , the three unit vectors (1, 0, 0), (0, 1, 0), (0, 0, 1) in the direction of three axes of a rectangular coordinate system form an orthonormal set.

Space l^2. In the space l^2, an orthonormal sequence is (e_n), where $e_n = (\delta_{nj})$ has the *n*th element 1 and all others zero.

Total orthonormal set

A *total* set (or *fundamental set*)in a normed space ***X*** is a subset ***M*** $\subset$ ***X*** whose span is dense in ***X.*** Accordingly, an orthonormal set (or sequence or family) in an inner product space which is total in ***X*** is called a total orthonormal set (or sequence or family, respectively) in ***X.***

M is total in ***X*** if and only if

$$\overline{\text{span}\, \boldsymbol{M}} = \boldsymbol{X}$$

This is obvious from the definition.

A total orthonormal family in ***X*** is sometimes called an *orthonormal basis* for ***X***. However, it is important to not that this is not a basis, in the sense of algebra, for ***X*** as a vector space, unless ***X*** is finite dimensional.

In every Hilbert space H≠ $\{0\}$ *there exists a total orthonormal set.*

Chapter 3

SELF-ADJOINT,UNITARY AND NORMAL OPERATORS

Hilbert- adjoint operator T*

Let T: $H_1 \to H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert spaces. Then the *Hilbert-adjoint operator T** of T is the operator

$$T^* : H_2 \to H_1$$

such that for all x $\in H_1$, and y $\in H_2$,

(1) $$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

Theorem (Properties of Hilbert-adjoint operators).

Let H_1, H_2 be Hilbert spaces, S: $H_1 \to H_2$ and T: $H_1 \to H_2$ bounded linear operators and α any scalar. Then we have

(a) $\langle T^* y, x\rangle = \langle y, Tx\rangle$ $(x \in H_1,\ y \in H_2)$

(b) $(S + T)^* = S^* + T^*$

(c) $(\alpha T)^* = \overline{\alpha}T^*$

(6) (d) $(T^*)^* = T$

(e) $\| T^*T \| = \| TT^*\| = \|T\|^2$

(f) $T^*T=0 \iff T=0$

(g) $(ST)^* = T^*S^*$ (assuming $H_2=H_1$).

Definition

An operator T $\in \mathcal{B}$(H) is said to be

(a) normal if $TT^* = T^*T$,

(b) self- adjoint (or hermitian) if $T^* = T$,

(c) unitary if $T^*T = I = TT^*$, where I is the identity operator on H,

(d) a projection if $T^2 = T$.

Theorem

An operator $T \in \mathcal{B}(H)$ is normal if and only if

(1) $$\| Tx \| = \| T^*x \|$$

for every $x \in H$. Normal operators T have the following properties:

(a) $\mathcal{N}(T) = \mathcal{N}(T^*)$.

(b) $\mathcal{R}(T)$ is dense in H if and only if T is one-to-one.

(c) T is invertible if and only if there exists $\delta > 0$ such that $\| Tx \| \geq \delta \| x \|$ for every $x \in H$.

(d) If $Tx = \alpha x$ for some $x \in H$, $\alpha \in \mathbb{C}$, then $T^*x = \bar{\alpha}x$.

(e) If α and β are distinct eigen values of T, then the corresponding eigen spaces are orthogonal to each other.

Theorem (Self-adjointness of product)

The product of two bounded self-adjoint linear operators S and T on a Hilbert space H is self-adjoint if and only if the operators commute,

$$ST = TS.$$

Proof

By property of Hilbert-adjoint operators and by the assumption,

$$(ST)^* = T^*S^* = TS.$$

Hence

$$ST = (ST)^* \quad \Leftrightarrow \quad ST = TS.$$

This completes the proof.

Theorem (Unitary operator)

Let the operators U: H→H and V: H→H be unitary; here, H is a Hilbert space. Then:

(a) U is isometric ; thus $\|Ux\| = \|x\|$ for all $x \in H$;

(b) $\|U\| = 1$, provided $H \neq \{0\}$,

(c) U^{-1}(=U*) is unitary,

(d) UV is unitary,

(e) U is normal.

Chapter 4

APPLICATIONS OF INNER PRODUCT SPACE

This section is important for applications of the geometric properties of an inner product space. However, it can be skipped without affecting our development of the theory. Here we deal with some problems of approximation which have a bearing on optimization subject to certain constraints. We consider the following question. What best can be done if we want to come close to a given element of an inner product space *X* while having to remain in a given subset of *X*? To formalize our study, we introduce the following notion.

1. Let **X** be an inner product space and **E** be a subset of **X**. Given an element x of X, an element y of E is said to be a best approximation from E to x if ||x - y|| ≤ ||x - z|| for all z € E, that is, ||x - y|| = dist(x, E). Such an element y is also known as an optimal solution of the following problem:

‘ Minimize || x – z || , subject to z € E.’

Then x - y is known as an optimal error.

2. Find the linear or quadratic least squares approximation of a function.

Many problems in the physical sciences and engineering involve an approximation of a function f by another function g. If f is in C [a, b] (the inner product space of all continuous functions on [a, b]), then g is usually chosen from a subspace W of C [a, b].

In particular, to approximate the function

f(x) = e^x

You could choose one of the following forms of g.

1. Linear g(x) = $a_0 + a_1x$
2. Quadratic g(x) = $a_0 + a_1x + a_2x^2$

Definition of Least Squares Approximation

Let f be continuous on [a, b] and let W be a subspace of C [a b]. A function g in W is called a **least squares approximation** of f with respect to W when the value of

I = $\int_a^b [f(x) - g(x)]^2\, dx$

is a minimum with respect to all other functions in W.

Note that if the subspace W in this definition is the entire space C [a, b], then g(x) = f(x), which gives I = 0.

3. Quadratic Forms.

Definition

A real quadratic form q in n variables x1...., xn is a polynomial such that every term has degree 2.

That is.

q$(x_1 + x_2 + \cdots x_n)$ = $\sum_i c_i x_i^2 + \sum_{i<j} d_{ij} x_i x_j$

Where c_i ϵ R, d_{ij} ϵ R for every i, j = 1.2.....n

the quadratic form q defines a symmetric matrix A = $[a_{ij}]$, where a_{ij} = c_i, and $a_{ij} = a_{ji} = \frac{1}{2} d_{ij}$.

If the matrix A of q is diagonal, then q has the diagonal representation

q(X) = X^T AX = $a_{11}x^2 + a_{22}x^2 \ldots + a_{nn}x^2$

That is, the quadratic polynomial representing q will contain no "cross product" terms.

Example1

If q(x_1, x_2) =$5x_1^2 - 8x_1x_2 + 9x_2^2$, Express q in matrix form.

Solution

Let A = $[a_{ij}]$, where a_{ij} is the coefficient of $x_i x_j$, then,

q(x_1, x_2) = $[x_1, x_2]\begin{bmatrix} 5 & -4 \\ -4 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Example2

If q(x_1, x_2, x_3) = $4x_1^2 - 8x_1x_2 + 6x_2x_3 + 2x_1x_3 + 3x_2^2 + 5x_3^2$, Express q in matrix form.

Solution

Let A = $[a_{ij}]$, where, a_{ij} is the coefficient ofx_ix_j, then,

$$q(x_1, x_2, x_3) = [x_1, x_2, x_3] \begin{bmatrix} 4 & 4 & 1 \\ -4 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

4. Some of the main ones are vectors in the Euclidean space and the Frobenius inner product for matrices.

5. Other than that, there are a lot of applications in Fourier analysis. Inner product spaces can be used to define Fourier coefficients for the series and the gives us a wide range of applications in boundary value problems.

CONCLUSION

This project gives a brief overview of the subject ,precisely some theoretical results. The main topic under this project is Inner product space.

In chapter 1, we show some important properties of inner product spaces. In chapter 2, we focus on orthogonality and theorems regarding it. In chapter 3,we saw its adjoint operators and in chapter 4,we study some applications of inner product spaces.

We consider that inner product spaces are very appropriate to deal with real applications and this formal study can be useful in many contexts.

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