"APPLICATIONS OF DIFFERENTIAL EQUATIONS"

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DECLARATION

We hereby declare that this project entitled "**APPLICATIONS OF DIFFERENTIAL EQUATIONS**" is a bonafide record of work done by us under the supervision of Ms. ALAKA MOHAN, Guest Lecturer, Department of Mathematics, Bharata Mata College, Thrikkakara and the work has not previously formed by the basis for the award of any academic qualification, fellowship or other similar title of any other University or Board

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CERTIFICATE

This is to certify that the project entitled "**APPLICATIONS OF DIFFERENTIAL EQUATIONS**" submitted jointly by Anjali Anil, Azlamiya T A, Sifna Suneer, Yunas K A, Ziya-U-Zaman in partial fulfilment of the requirements for the B.Sc. Degree in mathematics is a bonafide record of the studies undertaken by them under my supervision at the Department of Mathematics, Bharata Mata College, Thrikkakara, during 2021 – 2022. This dissertation has not been submitted for any other degree elsewhere.

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CHAPTER-1

INTRODUCTION

A differential equation is a mathematical equation that relates some function with its derivatives.

In applications the functions usually represent physical quantities, the derivatives represent their rates of change and the equation defines a relationship between the two. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics and biology.

In pure mathematics, differential equations are studied from several different perspectives, mostly concerned with their solutions- the set of functions that satisfy the equation. Only the simplest differential equations are solvable by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form.

Many real-life problems in science and engineering, when formulated mathematically give rise to differential equation. The differential equation is generally used to express a relation between the function and its derivatives. In Physics and chemistry, it is used as a technique for determining the functions over its domain if we know the functions and some of the derivatives.

In this section, we look at different application of first order and second order differential equations. The order of a differential equation is defined to be that of the highest order derivative it contains. first order differential equations are an equation that contain only first derivative, and it has many applications in mathematics, physics, engineering and many other subjects. The application of first order differential equation in temperature have been studied the method of separation of variables Newton's law of cooling were used to find the solution of the temperature problems that requires the use of first order differential equation and these solutions are very useful in mathematics, biology, and physics especially in analysing problems involving temperature which requires the use of Newton's law of cooling.

In the following chapters, we are going to focus more in detail about the following applications of first order differential equations:

i. Newton's law of cooling
ii. Falling object
iii. Population growth and decay
iv. Mixture problem
v. Electric circuit
vi. Spread of epidemics
vii. Dynamics of tumour growth
viii. Drug distribution in human body

And applications of second order differential equations:

i. RLC circuit
ii. Damped spring mass system
iii. Simple harmonic motion
iv. Simple pendulum

CHAPTER-2

EQUATIONS OF FIRST AND SECOND ORDER DIFFERENTIAL EQUATIONS

2.1. DIFFERENTIAL EQUATION

A *differential equation* is an equation relating some function *f* to one more of its derivatives. The general form of such a differential equation is

$$p_n(x,y)\left(\frac{dy}{dx}\right)^n + p_{n-1}(x,y)\left(\frac{dy}{dx}\right)^{n-1} + \cdots + p_1(x,y)\frac{dy}{dx} + p_0(x,y) = 0$$

2.1.1. EXAMPLE

$$\frac{d^2f}{dx^2}(x) + 2x\frac{df}{dx}(x) + f^2(x) = \sin x$$

2.2. ORDINARY DIFFERENTIAL EQUATION

A differential equation is an *ordinary differential equation* if it involves a function of a single variable and the *ordinary derivatives* of that function.

2.3. ORDER OF DIFFERENTIAL EQUATION

An ordinary differential equation of *order n* is an equation involving an unknown function *f* together with its derivatives

$$\frac{df}{dx}, \frac{d^2f}{dx^2}, \ldots, \frac{d^nf}{dx^n}$$

2.4. FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

An equation is said to be *first-order linear* if it has the form

$$y' + a(x)y = b(x)$$

2.4.1. EXAMPLE

$$y' + 2xy = x$$

2.5. SEPARABLE EQUATIONS

A first-order ordinary differential equation is *separable* if it is possible, by elementary algebraic manipulation, to arrange the equation so that all the dependent variables (usually the *y* variable) are on one side and all the independent variables (usually the *x* variable) are on the other side.

2.5.1. EXAMPLE

Solve the equation $y - x\frac{dy}{dx} = y^2 + \frac{dy}{dx}$.

The given equation can be written as

$$y - y^2 = (x+1)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{y - y^2} = \frac{dx}{x+1}$$

$$\Rightarrow \frac{dy}{y(1-y)} = \frac{dx}{x+1}$$

$$\Rightarrow \left(\frac{1}{y} + \frac{1}{1-y}\right)dy = \frac{dx}{x+1}$$

Integrating both sides, we get

$$\ln y - \ln(1-y) = \ln(x+1) + c$$

$$\Rightarrow \ln\left(\frac{y}{1-y}\right) = \ln k(x+1) \qquad \text{taking } c = \ln k$$

$\Rightarrow y = k(1-y)(x+1)$ is the required solution.

2.6. EXACT EQUATIONS

A differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be exact if there exists a function $f(x,y)$ such that

$$df(x,y) = M(x,y)dx + N(x,y)dy$$

Where $M(x,y)$, $N(x,y)$ and $f(x,y)$ are continuous functions and have continuous first derivatives on some rectangle of the (x,y) plane.

2.6.1. EXAMPLE

Solve the equation $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

Here

$$M(x,y) = x^2 - 4xy - 2y^2$$

$$\Rightarrow \frac{\partial M}{\partial y} = -4x - 4y$$

And

$$N(x,y) = y^2 - 4xy - 2x^2$$

$$\Rightarrow \frac{\partial N}{\partial x} = -4y - 4x$$

Thus $\frac{\partial M}{\partial y} = \frac{dN}{dx}$, and hence the equation is exact.

Then $$f(x, y) = \int M dx + g(y) = \int (x^2 - 4xy - 2y^2) dx + g(y)$$

$$= \frac{x^3}{3} - 2x^2y - 2y^2x + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = -2x^2 - 4yx + \frac{dg}{dy}$$

$$\Rightarrow N(x, y) = -2x^2 - 4yx + \frac{dg}{dy}$$

$$\Rightarrow y^2 - 4xy - 2x^2 = -2x^2 - 4yx + \frac{dg}{dy}$$

$$\Rightarrow y^2 = \frac{dg}{dy} \Rightarrow g(y) = \frac{y^3}{3}$$

$$\Rightarrow \int y^2 dy = \frac{y^3}{3}$$

Thus, to get $g(y)$ we have to integrate with respect to y those terms in $N(x, y)$ which are free from x.

2.7. LINEAR EQUATIONS

A differential equation in which the dependent variable and its differential coefficients occur only in the first degree is called a *Linear Differential Equation*. Therefore, a first order linear differential equation is of the form

$$\frac{dy}{dx} + p(x)y = Q(x)$$

Where $p(x)$ and $Q(x)$ are functions of x only.

2.7.1. EXAMPLE

Solve $(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}$.

Rewriting this equation, we get

$$\frac{dy}{dx} + \frac{1}{1 + x^2} y = \frac{e^{\tan^{-1} x}}{1 + x^2}$$

$$\int p\, dx = \int \frac{1}{1 + x^2} dx = \tan^{-1} x$$

Thus, the solution is

$$ye^{\tan^{-1} x} = \int e^{\tan^{-1} x} \frac{e^{\tan^{-1} x}}{1 + x^2} dx + c$$

To integrate, put $\tan^{-1} x = t$. Then $\frac{dt}{dx} = \frac{1}{1+x^2}$

So $$ye^t = \int e^{2t}\, dt + c = \frac{1}{2}e^{2t} + c$$

Therefore

$$y = \frac{1}{2}e^{\tan^{-1} x} + ce^{-\tan^{-1} x}$$

Is the required solution.

2.8. HOMOGENOUS EQUATIONS

A function $g(x, y)$ of two variables is said to be *homogenous of degree* α, for α a real number, if

$$g(tx, ty) = t^{\alpha} g(x, y) \qquad \text{for all } t > 0.$$

2.8.1. EXAMPLE

Solve $(x + y)dx - (x - y)dy = 0$.

The above equation can be written in the form

$$\frac{dy}{dx} = \frac{x+y}{x-y} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$$

Putting $z = \frac{y}{x}$, hence $y = zx$ and

$$\frac{dy}{dx} = z + x \cdot \frac{dz}{dx}$$

We get

$$z + x\frac{dz}{dx} = \frac{1+z}{1-z}$$

$$x\frac{dz}{dx} = \frac{1+z^2}{1-z}$$

Or

$$\frac{1-z}{1+z^2}\, dz = \frac{dx}{x}$$

Integrating,

$$\int \frac{dz}{1+z^2} - \int \frac{zdz}{1+z^2} = \int \frac{dx}{x}$$

$$arc \tan z - \frac{1}{2}\ln(1+z^2) = \ln x + C$$

Putting $z = \frac{y}{x}$, the result is

$$arc\tan\frac{y}{x} - \ln\sqrt{x^2 + y^2} = C$$

Thus, we have expressed *y* explicitly as a function of *x*, all the derivatives are gone, and we have solved the differential equation.

2.9. INTEGRATING FACTOR

An *integrating factor* is a function that is chosen to facilitate the solving of a given equation involving differentials.

To solve a first-order linear equation

$$y' + a(x)y = b(x)\ ,$$

Multiply both sides of the equation by the "integrating factor" $e^{\int a(x)dx}$ and then integrate.

2.9.1. EXAMPLE

Solve the differential equation $x^2y' + xy = x^3$.

The above equation can be written as

$$y' + \frac{1}{x}y = x$$

Now $a(x) = 1/x$, $\int a(x)\,dx = \ln|x|$, and $e^{\int a(x)dx} = |x|$.

Multiplying the differential equation through by this factor. Thus,

$$xy' + y = x^2$$

$$(x \cdot y)' = x^2$$

Integrating,

$$\int (x \cdot y)'\,dx = \int x^2\,dx$$

$$x \cdot y = \frac{x^3}{3} + C$$

$$y = \frac{x^2}{3} + \frac{C}{x}$$

2.10. REDUCTION OF ORDER

2.10.1. DEPENDENT VARIABLE MISSING

If the dependent variable *y* is missing from our differential equation, we make the substitution $y' = p$. This entails $y'' = p'$. Thus, the differential equation is reduced to first order.

2.10.2. INDEPENDENT VARIABLE MISSING

If the variable x is missing from our differential equation, we make the substitution $y' = p$. This time the corresponding substitution for y'' will be a bit different. To wit,

$$y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy} \cdot p$$

This change of variable will reduce our differential equation to first order. In the reduced equation, we treat p as the dependent variable (or function) and *y* as the independent variable.

2.11. SECOND ORDER LINEAR DIFFERENTIAL EQUATION

An equation is said to be *second-order differential* equation with constant coefficients if it has the form

$$ay'' + by' + cy = d$$

Where a, b, c, d are constants

2.12. THE METHOD OF UNDETERMINED COEFFICIENTS

The *method of undetermined coefficients* is an approach to finding a particular solution to certain nonhomogeneous ordinary differential equations and recurrence relations.

2.13. THE METHOD OF VARIATION OF PARAMETERS

Variation of parameters is a method for producing a *particular solution* to a nonhomogeneous equation by exploiting the solutions to the associated homogeneous equation.

CHAPTER- 3

APPLICATIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS

3.1. NEWTON'S LAW OF COOLING

Under certain conditions, the temperature rate of change of a body is proportional to the difference between the temperature T of the body and the temperature T_0 of the surrounding medium. This is known as ***Newton's Law of Cooling***. Here, we shall consider the case in which T_0 remains constant and also suppose that heat flows rapidly enough that the temperature T of the body is the same at all points of the body at a given time t

If T= f(t) denotes the temperature of the body at time t, then f satisfies the differential equation

$$\frac{dT}{dt} = k(T - T_0) \qquad (1)$$

Where k < 0, This equation can be solved by separating the variables

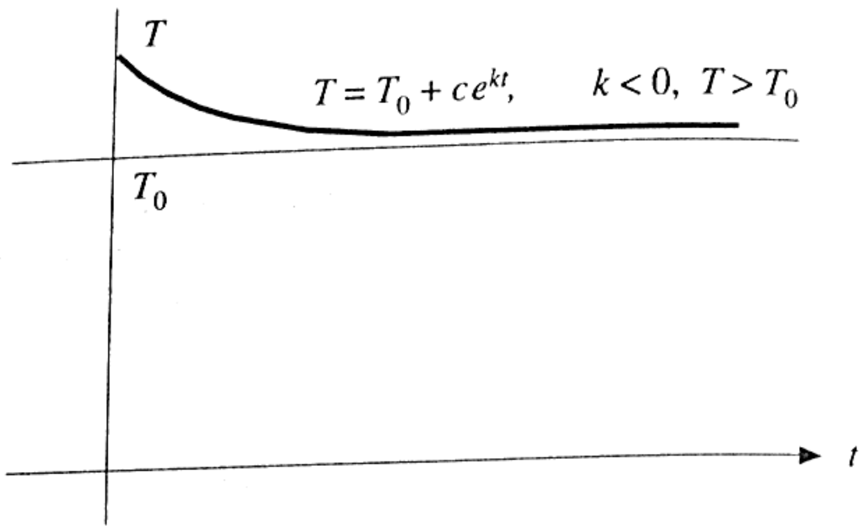
$$\frac{dT}{T - T_0} = k\,dt$$

Integrating both sides,

$$\text{Log}(T - T_0) = kt + C$$

$$T - T_0 = e^{kt+C} = e^{kt} \cdot e^{C} = C \cdot e^{kt}$$

$$T - T_0 = ce^{kt}T = T_0 + ce^{kt}, k < 0, \qquad T > T_0$$



Newton's law of cooling

3.1.1. PROBLEM

A body whose temperature T is initially 200^0C is immersed in a liquid when temperature T_0 is constantly 100^0C. If the temperature of the body is 150^0C at t = 1 minute, what is its temperature at t = 2 minutes?

SOLUTION

Separating the variables in Eq. (1), we get

$$\frac{dT}{T-100} = kdt$$

And the solution is

$$\log(T-100) = kt + C \qquad (2)$$

When t = 0, T = 200, we find that C = log 100. Also, at t = 1, T = 150 and

Eq. (2) gives

$$\log 50 = k(1) + log\, 100$$

Or

$$k = -\log(2)$$

Now, substituting C = log 100 and k = $-\log(2)$in Eq. (2), we obtain

$$\text{Log}(T-T_0) = -t\log(2) + \log(100)$$

Or

$$T = 100[1+2^{-t}]$$

Thus, at t = 2min, T = 125^0C.

Remark:

Consider Eq. (1) which has a solution of the form

$$\log(T-T_0) = kt + \log(C)$$

Or

$$T = T_0 + ce^{kt}$$

Now as $t \to \infty$, then $T \to T_0$ and consequently,

$$\frac{dT}{dt} = k(T - T_0) \to 0$$

That is, as *t* becomes large, the difference between temperature of the body and the temperature of the surrounding medium approaches zero, and the rate at which the body cools also approach zero.

3.2. FALLING OBJECT

An object is dropped from a height at time t = 0. If $h\ (t)$ is the height of the object at time t, a(t) the acceleration and $v(t)$ the velocity.

The relationships between a, v and h are as follows:

$$a\ (t) = \frac{dv}{dt}\ , v\ (t) = \frac{dh}{dt}$$

For a falling object $a\ (t)$ is constant and is equal to g = $-9.8\ m/s$

Combining the above differential equations, we can easily deduce the following equation

$$\frac{d^2h}{d^2t} = g$$

Integrate both sides of the above equation to obtain

$$\frac{dh}{dt} = gt + v_0$$

Integrate one more time to obtain

$$h(t) = \frac{1}{2}gt^2 + v_0\,t + h_0$$

The above equation describes the height of a falling object, from an initial height h_0 at an initial velocity $v_{0,}$ as a function time.

3.2.1 PROBLEM

An object falling in a vacuum subject to a constant gravitational force accelerates at a constant rate. If the object were to be dropped from rest and to attain a velocity of 5m/s after one second, how fast would it be travelling after five seconds?

SOLUTION

Let $v\ (t)$ be the velocity at time t seconds measured in meters per seconds. Then we know that $v\ (0)\ =\ 0$, that $v\ (1\ )\ =\ 5$, and that $v''=\ 0$ (the acceleration, the rate of change of velocity, so v', is a constant)

Integrating the equation $v''=\ 0$ with respect to t, we see that

$$v'\ (t)\ -\ v'(0)\ = 0$$

Thus, if $c_1\ =\ v'\ (0)$, we have $v\ (t)\ =\ c_1$

Integrating again, we see that,

$$v(t)\ -\ v(0)\ =\ c_1\ t$$

Setting $c_2\ =\ v\ (0)$, we have $v\ (t)\ =\ c_2\ +\ c_1\ t$

Evaluating at 0 and 1, we have,

$$0\ =\ v\ (0)\ =\ c_2\ +\ c_1(0)\ =\ c_2$$

And $$5\ =\ v\ (1)\ =\ c_2\ +\ c_1(1)\ =\ 0\ +\ c_1\ =\ c_1$$

Thus, $v\ (t)\ =\ 5t$ so that $v\ (5)\ =\ 25$

3.3. POPULATION GROWTH AND DECAY

When a population grows exponentially, it grows at a rate that is proportional to its size at any time t.

Let $p\ (t)$ be the number of individuals in a population at time t. The population will change with time t = 0

$$P\ (0) =\ P_0$$

Indeed, the rate of change of p will be due to births and deaths.

Rate of change of P = Rate of births − Rate of deaths

Assume that all individuals are identical in the population, and that average per capita birth rate, r, and that the average per capita mortality rate, m are some fixed positive constants

Then the total number of births into the population in year *t* is *rp*, and the total number of deaths out of the population in year *t* is *mp*. The rate of change of population as a whole is given by the derivative $\frac{dp}{dt}$

Thus,

$$\frac{dp}{dt} = rP - mP = (r-m)P = kP, \qquad where\ k = (r-m)$$

This means that the population satisfies a differential equation of the form

$$\frac{dp}{dt} = kp \qquad (1)$$

provided k is the so-called “not growth rate”,

That is, the birth rate minus mortality rate and is a constant

i.e.

$$k = \frac{1}{p} \cdot \frac{dp}{dt} \qquad \text{From (1)}$$

We can solve equation (1) using separation of variables

i.e.

$$\frac{dp}{p} = k\ dt$$

Integrating both sides

$$\int \frac{dp}{p} = \int k\ dt$$

$$\text{Ln}\,|\,p\,| = kt + C$$

Taking exponential

$$e^{\ln|p|} = e^{kt + c}$$

$$|\, p \,| \;=\; e^{kt} \cdot e^{C}$$

$$p\,(t) \;=\; Ae^{kt}$$

Using absolute value definition and by replacing constant with A

$$P\,(t) \;=\; Ae^{kt}$$

Which is the general solution of the differential equation

Using initial condition $P\,(0) \;=\; P_0$, we can find the particular solution

$$P_0 \;=\; P(0) \;=\; Ae^{k\cdot 0}$$
$$P_0 \;=\; A \cdot 1$$
$$P_0 \;=\; A$$

Hence, $P(t) \;=\; P_0\, e^{kt}$ is a particular solution;

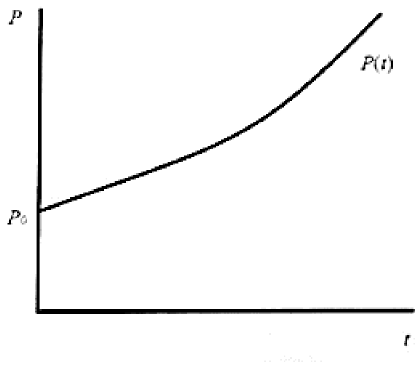
Where, P_0 = initial population at time t = 0,

k = relative growth rate that is constant,

t = the time the population grows,

$P(t)$ = what the population grows to after time t

The graph of the exponential equation $P(t) \;=\; P_0\, e^{kt}$ has the general form



3.3.1 PROBLEM

The population of a community is known to increase at a rate proportional to the number of people present at a time t. If the population has doubled in 6 years, how long will it take to triple?

SOLUTION

Let P denote the population at time t. Let $P(t)$ denote the initial population (population at t = 0)

The solution of the model

$$\frac{dp}{dt} = kP \text{ is}$$

$$P = Ae^{kt}, \text{ where } A = P_0 \text{ by given data}$$

$$Ae^{6k} = P(6) = 2P(0) = 2A$$

Or

$$e^{6k} = 2 \text{ or } K = \frac{1}{6}$$

Find t when $P(t) = 3A = 3P(0)$

Or

$$P(0)e^{kt} = 3P(0)$$

$$3 = e^{\left(\frac{1}{6}\right)\ln(2)t}$$

Or

$$\ln(3) = \frac{\ln(2)t}{6}$$

Or

$$t = \frac{6\ln(3)}{\ln(2)} = 9.6 \text{ years approximate.}$$

3.4. MIXTURE PROBLEM

Consider a tank containing G_0 gallons of solution in which P_0 lb of a substance S is dissolved. A second solution flows into the tank at a given rate r_1 gal/min, this solution containing P_1 lb/gal of S.

Finally, the mixture in the tank flows out at a given rate r_2 gal/min. Find out the number of pounds P of S in the tank at time t > 0.

Assume that the mixture in the tank is well-stirred, so that at any given time $t_k, P\ (t)\ =\ P\ (t_k)$ has the same value at each point in the tank. The rate at which P changes with time t is

$$\frac{dp}{dt} = (rate\ S\ flowin) - (rate\ S\ flowout) \qquad (1)$$

Equation (1) is called equation of continuity. It states that the mass of the quantity S is conserved. i.e., no amount of S is created or destroyed in the process.

The rate at which S flows into the tank is $P_1 r_1$ lb/min. The rate at which S flows out in the tank is $[P/G(t)]r_2$ lb/min., where $[P/G(t)]$ is the concentration of S in the tank at the time t; $G(t)$ is the number of gallons in the tank at time t. If $r_1\ = r_2$, $G(t)$ will have the constant value $G_0(t)$. Many interesting cases arise, each leading to a different differential equation, e.g., either $r_1\ or\ r_2$ maybe zero, $P_0\ or\ P_1$ ma be zero and so on

3.4.1. PROBLEM

A tank contains 100 gallons of brine in which 10lb of salt are dissolved. Brine containing 2 lb salt per gallon flows into the tank at 5 gal/min. If the well-stirred mixture is drawn off at 4 gal/min., find: (a) the amount of the salt in the tank at time t, and (b) the amount of the salt in the tank at t = 10 min.

SOLUTION

Let $P\ (t)$ denote the number of pounds of salt in the tank and $G\ (t)$ the number of gallons of brine at time t. Then $G\ (t) = 100 + t$. Also, $P\ (0) = 10$ and $G\ (0) = 100$ since 5(2) = 10 lb salt is added to the tank per minute and $[P/(100\ +\ t)](4)$ lb salt per minute is extracted from the tank, P then satisfies the differential equation

$$\frac{dP}{dt} = 10 - \frac{4P}{100 + t} \qquad (2)$$

Equation (2) can be written as

$$\frac{dP}{dt} + \frac{4P}{100\ +\ t} = 10$$

Which is a linear equation and the solution is

$$P(100\ +\ t)^4 = \int 10(100\ +\ 4)^4 dt = 2(100\ +\ t)^5 + C$$

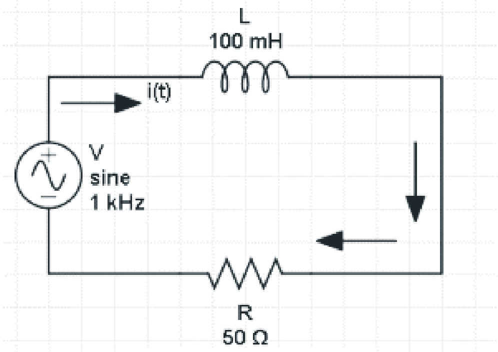
Putting t = 0 and P = 10, we get c $= -190(100)^4$. Thus

(a) $P(t) = 2(100 + t) - 190(100)^4(100 + t)^{-4}$

(b) $P(10) = 2(100 + 10) - 190(100)^4 + (100 + 10)^{-4} = 90.2\, lb$

3.5. ELECTRIC CIRCUITS

Let a series circuit contain only a resistor and an inductor as shown in the fig.

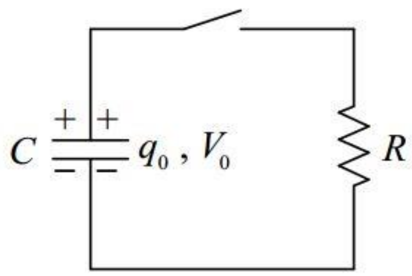


LR Series Circuit

By Kirchhoff's second law the sum of the voltage drop across the inductor $\left(L\,\frac{di}{dt}\right)$ and the voltage drop across the resistor (iR) is the same as the impressed voltage $(E(t))$ on the circuit. Current at time t, $i(t)$, is the solution of the differential equation

$$L\frac{di}{dt} + Ri = E(t) \qquad (1)$$

Where L and R are constants known as the inductance and the resistance respectively.



RC Series Circuit

The voltage drop across a capacitor with capacitance C is given by q(t)/C, where q is the charge on the capacitor. Hence, for the series circuit shown in Fig. we get the following equation by applying Kirchhoff's second law:

$$Ri + \frac{1}{c}q = E(t) \qquad (2)$$

Since $i = \frac{dq}{dt}$, Eq. (2) can be written as

$$R\frac{dq}{dt} + \frac{1}{c}q = E(t) \tag{3}$$

3.5.1. PROBLEM

Find the current in a series RL circuit in which the resistance, inductance, and voltage are constant. Assume that $i(0) = 0$ i.e., initial current is zero.

SOLUTION

It is modelled by Eq. (1)

$$L\frac{di}{dt} + Ri = E(t)$$

Or

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E(t)}{L} \tag{4}$$

Since L, R and E are constants,

Equation (4) is a linear differential equation of first order in i with integrating factor

$$e^{\int\frac{R}{L}dt} = e^{Rt/L}$$

The solution of Eq. (4) is

$$i(t)e^{\frac{Rt}{L}} = \int \frac{E}{L}e^{Rt/L}\,dt$$

$$i(t)e^{\frac{R}{L}t} = \frac{E}{L}e^{Rt/L}\frac{L}{R} + c$$

Or

$$i(t) = \frac{E}{R} + ce^{-\frac{R}{L}t} \tag{5}$$

since

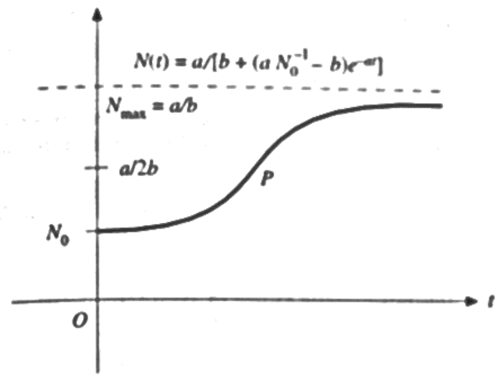
$$i(0) = 0, c = -\frac{E}{R}$$

Putting this value of c in Eq. (5) we get

$$i(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

3.6. SPREAD OF EPIDEMICS

An important problem in biology and medicine deals with the occurrence spreading and control of a contagious disease, i.e., one which can be transmitted from one individual to another. The science that deals with this study is called *epidemiology*, and if a large number of population gets the disease, we say that there is *epidemic*.



Logistic curve.

To have a simple mathematical description for the spread of a disease suppose that there is a large but finite population. Let us restrict ourselves to the students in some large college or university, who remain on campus for a relatively long period and do not have excess to other communities. We presuppose that there are only two types of students, those who have a contagious disease (called infected), and those who do not have the disease, (i.e., unaffected) but are capable of contracting it on exposure to an infected student. If there are some infected students initially, then we want to find a formula for the number of infected students at any time.

Let N_i denote the number of infected students at any time t and N_u the uninfected students. Then, if N is the total number of students (assumed to be constant), we have

$$N = N_i + N_u \qquad (1)$$

Here, dN_i/dt is the time rate of change in the number of infected students and should depend in some way on N_i, and thus N_u. Assuming that dN_i/dt is the quadratic function of N_i as an approximation, we get

$$\frac{dN_i}{dt} = a_0 + a_1 N_i + a_2 N^2{}_i \qquad (2)$$

where a_0, a_1, a_2 are constants. Now we would expect $\frac{dN_i}{dt} = 0$, where$N_i = 0$, i.e., there are no infected students and where $N_i = N$, i.e., all students are infected. Then from Eq. (2), we have

$$a_0 = 0 \qquad \text{and} \qquad a_1 N + a_2 N^2 = 0 \qquad \text{or} \qquad a_2 = -a_1/N$$

So that Eq. (2) becomes

$$\frac{dN_i}{dt} = a_1 N_i - \frac{a_1 {N_i}^2}{N} = \frac{a_1}{N} N_i (N - N_i)$$

$$= k N_i (N - N_i)$$

where $k = a_1/N$ and the initial conditions are

$$N_i = N_0 \qquad \text{at} \qquad t = 0$$

The Eq. (3) and (4) has the solution as

$$N_i = \frac{N}{1 + \left(\frac{N}{N_0} - 1\right) e^{-kNt}} \tag{5}$$

The graph of Eq. (5) is the logistic curve. From the shape of the logistic curve, we see that initially there is a gradual increase in the number of infected students, followed by a rather sharp rise in their number near the infected point, and finally a tapering off. The limiting case occurs where all students become infected, as seen by Eq. (5). That $N_i \to N$ as $t \to \infty$.

3.6.1. PROBLEM

Spread of flu virus. A student carrying a flu virus returns to an isolated college hostel of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number N_i of infected students but also to the students not infected. Find the number of infected students after 6 days when it is further observed that after 4 days $N_i(4) = 50$.

SOLUTION

Assuming that no one leaves the hostel throughout the duration of the duration of the disease, we must then solve the initial value problem

$$\frac{dN_i}{dt} = kN_i(N - N_i) \qquad N(0) = 1 = kN_i(1000 - N_i),$$

We have, from Eq. (5)

$$N_i = N(t) = \frac{1000}{1 + 999e^{-1000kt}} \tag{6}$$

Now, using $N_i = N(4) = 50$, we can determine k.

$$50 = \frac{1000}{1 + 999e^{-1000k \times 4}}$$

Or

$$e^{-4000k} = \frac{19}{999}$$

Or

$$k = 0.0009906$$

Thus, Eq. (6) becomes

$$N_i = N(t) = \frac{1000}{1 + 999e^{-0.0009906t}}$$

Or

$$N_i = N(6) = \frac{1000}{1+999e^{-5.9436}} = 276 \text{ students}$$

3.7. DYNAMICS OF TUMOUR GROWTH

It has been observed experimentally that free-living dividing cells, such as bacteria cells, grow at a rate proportional to the volume of dividing cells at that moment.

Let $V(t)$ denote the volume of the dividing cells at the time t. Then

$$\frac{dV}{dt} = kV \tag{1}$$

for some positive constant k. The solution of Eq. (1) is

$$V(t) = V_0 e^{k(t-t_0)} \tag{2}$$

Where V_0 is the volume of dividing cells at time t_0 (initial time). Thus, free-living dividing cells grow exponentially with time, whereas solid tumours do not grow exponentially with time. As the tumour becomes larger, the doubling time of the total tumour volume continuously increases. Almost a thousand-fold increase in tumour volume, by the equation

$$V(t) = V_0\, exp\left[\frac{k}{a}(1 - e^{-at})\right] \tag{3}$$

Where k and a are positive constants.

Equation (3) is usually known as a *Gompertzian relation*. It states that tumour grows more and more slowly with the passage of time, and that it ultimately approaches the limiting volume $V_0 e^{k/a}$. An insight into this problem can be gained by finding a differential equation satisfied by $V(t)$.

Differentiation of Eq. (3) yields

$$\frac{dV}{dt} = V_0 k e^{-at} e^{\left[\frac{k}{a}(1-e^{-at})\right]} = k e^{-at} V \tag{4}$$

Equation (4) can also be arranged as

$$\frac{dV}{dt} = (k e^{-at})V \tag{4a}$$

$$\frac{dV}{dt} = k(e^{-at}V) \tag{4b}$$

With these arrangements of Eq. (4), two theories have been evolved for the dynamics of tumour growth. According to the first theory, the retarding effect of tumour growth is due to an increase in the mean generation time of the cells, without a change in the proportion of the reproducing cells. As time goes on, the reproducing cells mature or age, and thus divide more slowly. This theory corresponds to Eq. (4a). On the other hand, the second theory corresponding to Eq. (4b) is the mean generation time of the dividing cells remains constant, and the retardation of growth is due to a loss in reproductive cells in the tumour.

Possible explanation for this is that a necrotic region develops in the center of the tumour. This necrosis appears at a critical size for a particular type of tumour, and thereafter the necrotic “core” increases rapidly as the total tumour mass increases. According to this theory a necrotic core develops because in many tumours the supply of blood, and thus of oxygen and nutrients, is almost confined to the surface of the tumour and a short distance beneath it. As the tumour grows, the supply of oxygen to the central core by diffusion becomes more and more difficult resulting in the formation of a necrotic core.

3.8. DRUG DISTRIBUTION (CONCENTRATION) IN HUMAN BODY

To combat the infection to a human body, appropriate dose of medicine is essential. Because the amount of the drug in the human body decreases with time, medicine must be given in multiple doses. The rate at which the level *y* of the drug in a patient's drug decays can be modeled by the decay equation

$$\frac{dy}{dt} = -ky$$

Where *k* is a constant to be experimentally determined for each drug. If initially, i.e., at $t = 0$ a patient is given an initial dose y_p, then the drug level *l* at any time *t* is the solution of the above differential equations. i.e.

$$y(t) = y_p e^{-kt}$$

Remark:

In this model it is assumed that the ingested drug is absorbed immediately, which is not usually the case. However, the time of absorption is small compared with the time between doses.

3.8.1. PROBLEM

A representative of a pharmaceutical company recommends that a new drug of his company be given every *T* hours in doses of quantity y_0, for an extended period of time. Find the steady state drug in the patient's body.

SOLUTION

Since the initial dose is y_0, the drug concentration at any time $t \geq 0$ is found by the equation $y = y_0 e^{-kt}$, the solution of the equation $dy/dt = -ky$.

At $t = T$ the second dose of y_0 is taken, which increases the drug level to

$$y(T) = y_0 + y_0 e^{-kT} = y_0(1 + e^{-kT})$$

The drug level immediately begins to decay. To find its mathematical expression we solve the initial-value problem:

$$\frac{dy}{dt} = -ky$$

$$y(T) = y_0(1 + e^{-kT})$$

Solving this initial-value problem we get

$$y = y_0(1 + e^{-kT})e^{-k(t-T)}$$

This equation gives the drug level for $t > T$. The third dose of y_0 is to be taken at $t = 2T$ and the drug just before this dose is taken is given by

$$y = y_0(1 + e^{-kT})e^{-k(2T-T)} = y_0(1 + e^{-kT})e^{-kT}$$

The dosage y_0 taken at $t = 2T$ raises the drug level to

$$y(2T) = y_0 + y_0(1 + e^{-kT})e^{-kT} = y_0(1 + e^{-kT} + e^{-2kT})$$

Continuing in this way, we find after (n+1)th dose is the sum of the first n+1 terms of a geometric series, with first term as y_0 and the common ratio e^{-kT}. This sum can be written as

$$y(nT) = \frac{y_0(1 - e^{-(n+1)})}{1 - e^{-kT}}$$

As n becomes large, the drug level approaches a steady-state value, say y_s given by

$$y_s = \lim_{n\to\infty} y(nT)$$

$$= \frac{y_0}{1 - e^{-kT}}$$

The steady-state value y_s is called the saturation level of the drug.

CHAPTER- 4

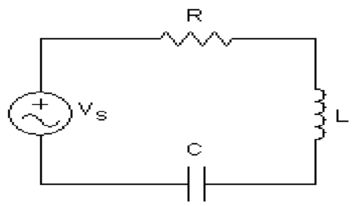
APPLICATIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS

4.1. RLC CIRCUIT

The electric circuit with current I in amperes satisfies the differential equation

$$L\frac{dI}{dt} + RI = E(t)$$

The figure shown contains an additional element known as *capacitor*. This type of element stores electrical energy in the circuit.



Circuit diagram consisting of a capacitor C, a resistor R and an inductor L

The voltage drop across a capacitor is proportional to the charge q (in coulombs) on the capacitor and is given by $C^{-1}q$, where C^{-1} is the constant of proportionality. The constant C is called the *constant of capacitance* or simply *capacitance*. Apply Kirchhoff 's law to fig., we get the differential equation

$$L\frac{dI}{dt} + RI + \frac{1}{C}q = E(t) \tag{5}$$

The current I equals the time rate of change of q, i.e.

$$I(t) = \frac{dq(t)}{dt} \tag{6}$$

From Eqs. (1) and (2)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t) \tag{7}$$

Differentiating both sides of Eq. (1) w.r.t. I and using Eq. (2), we obtain

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{d}{dt}E(t) \tag{8}$$

We assume here that L (H), R (Ω) and C (F) are constants and $E(t)$ (V) is the impressed voltage and t (s) is the time.

4.1.1. PROBLEM

A series circuit contains only a capacitor and inductor. If the capacitor has an initial charge q_0, determine the subsequent charge $q(t)$.

SOLUTION

Form Eq. (3),

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$$

and the given initial conditions are $q(0) = q_0$. Assume that no current flows initially; then $q'(0) = 0$, since $q'(t) = I(t)$. The general solution is, thus

$$q(t) = c_1 \cos\frac{1}{\sqrt{LC}}t + C_2 \sin\frac{1}{\sqrt{LC}}t$$

The initial conditions imply that $c_1 = q_0$ and $c_2 = 0$, so that

$$q(t) = q_0 \cos\frac{1}{\sqrt{LC}}t$$

4.2. DAMPED SPRING MASS SYSTEM

Enumerating the forces acts upon the mass. Forces tending to pull the mass downward are positive, while those tending to pull it upward are negative. The forces are:

1. F_1, the *force of gravity*, of magnitude mg, where g is the acceleration due to gravity. Since this act in the downward direction, it is positive, and so

$$F_1 = mg \quad (1)$$

2. F_2, the *restoring force* of the spring. Since $x + l$ is the total amount of elongation, by Hook's law the magnitude of this force is $k(x + l)$. When the mass is below the end of the unstretched spring, this force acts in the upward direction and so is *negative*. Also, for the mass in such a

position, $x + l$ is *positive*. Thus, when the mass is *below* the end of the unstretched spring, the restoring force is given by

$$F_2 = -k(x + l) \qquad (2)$$

This also gives the restoring force when the mass is *above* the end of the unstretched spring. When the mass is at rest in its equilibrium position the restoring force F_2 is equal in magnitude but opposite in direction to the force of gravity and so is given by $-mg$. Since in this position $x = 0$, Equation (2) gives

$$-mg = -k(0 + l)$$

or

$$mg = kl$$

Replacing kl by mg in Equation (2) we see that the restoring force can thus be written as

$$F_2 = -kx - mg \qquad (3)$$

3. F_3, the *resisting force* of the medium, called the *damping force*. Although the magnitude of this force is not known *exactly*, it is known that for small velocities it is *approximately* proportional to the magnitude of the velocity:

$$|F_3| = a\left|\frac{dx}{dt}\right| \qquad (4)$$

Where a > 0 is called the *damping constant*. When the mass is moving *downward*, F_3 acts in the *upward* direction (opposite to that of the motion) and so $F_3 < 0$. Also, since m is moving *downward*, x is *increasing* and $\frac{dx}{dt}$ is *positive*. Thus, assuming Equation (4) to hold, when the mass is moving *downward*, the damping force is given by

$$F_3 = -a\frac{dx}{dt} \quad (a > 0) \qquad (5)$$

This also gives the damping force when the mass is moving *upward*.

4. F_4, any *external impressed forces* that act upon the mass. Let us denote the resultant of all such external forces at time t simply by $F(t)$ and write

$$F_4 = F(t) \qquad (6)$$

We now apply Newton's second law, $F = ma$, where $F = F_1 + F_2 + F_3 + F_4$. Using (1), (3), (5), and (6), we find

$$m\frac{d^2x}{dt^2} = mg - kx - mg - a\frac{dx}{dt} + F(t) \qquad (7)$$

or

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = F(t) \tag{8}$$

This we take as the differential equation for the motion of the mass on the spring. Observe that it is a nonhomogeneous second-order linear differential equation with constant coefficients. If $a = 0$ the motion is called *undamped*; otherwise, it is called *damped*. If there are no external impressed forces, $F(t) = 0$ for all t and the motion is called *free*; otherwise, it is called *forced*.

4.3. SIMPLE HARMONIC MOTION

The motion of a particle moving in a straight line is described by the following differential equation:

$$F\left(t\,, x\,, \frac{dx}{dt}\,, \frac{d^2x}{dt^2} \right) = 0 \tag{1}$$

The general solution of Eq. (1) contains two arbitrary constants, which can be obtained from the initial conditions

$$t = 0, \qquad x = x_0, \qquad t = b, \qquad \frac{dx}{dt} = v_0 \tag{2}$$

The resulting particular solution is given by the displacement function $x = h(t)$. The domain of h is the time interval. The differential Eq. (1) and the initial conditions (2) furnish a mathematical model for the physical situation.

4.3.1. PROBLEM

A particle moves on the x-axis with an acceleration $a = 6t - 4$ ft/s^2. Find the position and velocity of the particle at $t = 3$, if the particle is at origin and has a velocity 10 ft/s when $t = 0$.

SOLUTION

Here, at $t = 0,\ x = 0,\ v = 10$.

Now,

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 6t - 4$$

which gives

$$v = 3t^2 - 4t + c_1$$

And thus $c_1 = 10$. Also

$$v = \frac{dx}{dt} = 3t^2 - 4t + 10 \tag{1}$$

Which on integration yields

$$x = t^3 - 2t^2 + 10t + c_2$$

From the initial conditions, $c_2 = 0$. Thus

$$x = t^3 - 2t^2 + 10t \tag{2}$$

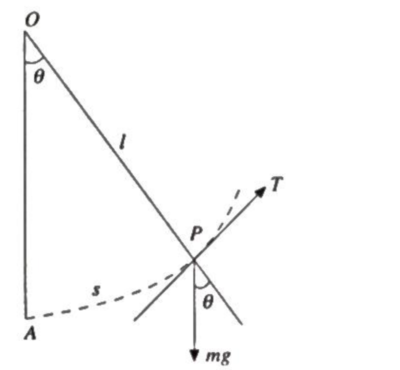
Putting $t = 3$ in Eqs. (1) and (2), we get

$$(v)_{t=3} = 25 \text{ ft/s}, \quad (x)_{t=3} = 39 \text{ ft}$$

4.4. SIMPLE PENDULUM

A simple pendulum consists of a particle of weights W (bob) supported by a straight rod or piece of string of length l. The particle is free to oscillate in a vertical plane; the mass of the particle is assumed to be concentrated at a point; and the weight of the rod is assumed to be negligible.

Let *0* be the fixed point and A be the position of the bob initially.



Schematic diagram for a simple pendulum.

If P is the position of the bob at any time t, such that $arc\ AP = s$ and $\angle AOP = \theta$, then $s = l\theta$.

Now, the equation of motion along PT is

$$m\frac{d^2s}{dt^2} = -mg\sin\theta$$

Or

$$\frac{d^2(l\theta)}{dt^2} + g\sin\theta = 0$$

Or

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

Or

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}(\theta - \frac{\theta^3}{3!} + \cdots) = 0$$

Or

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \text{ (For a first approx.)}$$

The auxiliary equation has the roots $\pm\sqrt{g/l}$, and the solution is

$$\theta = c_1 \cos\sqrt{\frac{g}{l}}t + c_2 \sin\sqrt{\frac{g}{l}}t$$

Therefore, the motion of a pendulum is simple harmonic and the time of an oscillation is $2\pi\sqrt{l/g}$.

The movement of the bob from one end to the other constitutes half an oscillation and is known as a *beat* or a *swing*. The time of one beat is $\pi\sqrt{l/g}$.

4.4.1. PROBLEM

A simple pendulum of length l is oscillating through a small angle θ in a medium for which the resistance is proportional to the velocity. Obtain the differential equation of its motion and discuss the motion.

SOLUTION

The equation of motion along the tangent PT is

$$m\frac{d^2s}{dt^2} = -mg\sin\theta - \lambda\frac{ds}{dt}$$

Where λ is a constant, or

$$\frac{d^2\theta}{dt^2} + g\sin\theta + \frac{\lambda}{m}\frac{d}{dt}(l\theta) = 0$$

Replace $\sin\theta$ by θ (as θ is small) and put $\frac{\lambda}{m} = 2k$, to obtain

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \frac{g\theta}{l} = 0$$

Which is the required differential equation.

The auxiliary equation has the roots $-k \pm \sqrt{W^2 - k^2}$, where $W = g/l$. The oscillatory motion of the bob is possible only when $k < W$. The solution of the present differential equation is

$$\theta = e^{-kt}\left(c_1\cos\sqrt{W^2 - k^2}t + c_2\sin\sqrt{W^2 - k^2}t\right)$$

Which gives the vibratory motion of period $2\pi/\sqrt{W^2 - k^2}$.

CHAPTER-5

CONCLUSION

Differential equations play major role in applications of sciences and engineering. it arises in wide variety of engineering applications for e.g., electromagnetic theory, signal processing, computational fluid dynamics, etc. These equations can be typically solved using either analytical or numerical methods. Since many of the differential equations arising in real life application cannot be solved analytically or we can say that their analytical solution does not exist.

Here in this book, we have started with the fundamental concept of differential equation, some real-life applications where the problem is arising and explanation of some existing methods for their solution. We have given a basic presentation of application of differential equations. We have learned about various application in real life and in mathematics along with its definition and its types.

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