REPRESENTATION THEORY OF FINITE GROUPS

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CERTIFICATE

This is to certify that the dissertation entitled **“REPRESENTATION THEORY OF FINITE GROUPS”** submitted by ROSEMARY BENNY is a record of work done by the candidate during her period of study under my supervision and guidance.

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DECLARATION

I hereby declare that the project report entitled **"REPRESENTATION THEORY OF FINITE GROUPS"** submitted to Mahatma Gandhi University , is a record of an original work done by me under the supervision of Dr. Lakshmi.C and the project has not formed the basis for the award of my academic qualification fellowship or other similar title of other university or board .

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CONTENTS

ABSTRACT

The Project Report on “Representation Theory of Finite Groups” consist of six chapters.

The first chapter helps us to recollect the basic definitions and examples about Groups and Homomorphisms. The Kernel of a Homomorphism is also discussed here.

The second chapter focuses on Linear Algrbra. The definitions and examples on Vector spaces and about Linear Transformations are explained here.

The concept about Group Represention is described in the third chapter. Here the title of the project is been introduced and it also provides us with some examples ,definition and theorems based on this topic.

In chapter 4 , we try to understand the concept of representation of Groups by using Linear Algebra. Here we study about FG Modules which will help us to understand the Representation Theory in more depth. It makes our study more easier.

Chapter 5 tells about FG submodules and irreducibility.

Finally the last chapter concludes the project by introducing the concept of Group Algebra. By defining Group Algebra of G it helps us to construct an important faithful representation of G known as the Regular representation of G.

In these chapters , we consider Group G as a Finite Group and V as a vector space over F, where F is either R or C .

INTRODUCTION

Representation theory is concerned with the ways of writing a group as a group of matrices . Otherwise we can simply say that it is a study of groups as a matrices. An attractive feature of representation theory is that it combines the main two strands of mathematics namely Group Theory and Linear Algebra. Not only about its beautiful theory , but it also provides one of the keys for a proper understanding of finite groups. For example , it is important to a concrete description about a particular group ; this is achieved by finding a representation of the group as a group of matrices. Moreover by studing the different representations of the group it is possible to prove results which lie outside the frame work of representation theory.

Infact, the range of applications of the theory extends far beyond the boundaries of pure mathematics and includes theoretical physics and chemistry.

CHAPTER 1

GROUPS AND HOMOMORPHISMS

GROUPS

1.1 Definition

A Group <G,*> is a set G, closed under a binary operation *,such that

→ For all a,b,c ∈ G, we have

$(a * b) * c = a * (b * c)$; Associativity of *

→ There is an element e in G such that for all x ∈ G

$e * x = x * e = x$; Identity element for *

→ Corresponding to each element a ∈ G, there is an element $a^{'}$ in G such that

$a * a^{'} = a^{'} * a = e$; Inverse $a^{'}$ of a

Note

If the number of elements in G is finite ,then we call G as a Finite Group. The number of elements in a Group G is called as the order of G and is denoted by $|G|$

1.2 Examples

1) Let n be a positive integer. Let C denote the set of all complex numbers. The set of n^{th} roots of unity in C, under usual multiplication of complex

numbers, forms a group of order n. It is written as C_n and is called as the Cyclic Group of order n. If a $=e^{\frac{2\pi i}{n}}$ then

$$C_n = \{1, a, a^2, \; .... \; a^{n-1}\} \text{ and } a^n = 1$$

2) Let n be an integer with n ≥ 3, and consider the rotation and reflection symmetries of a regular n-sided polygon. There are n rotation symmetries: these are $\rho_0, \rho_1 . . . , \rho_{n-1}$ where ρ_k is the (clockwise) rotation about the centre O through an angle $2\pi k/n$.

There are also n reflection symmetries: these are reflections in the n lines passing through O and a corner or the mid-point of a side of the polygon.

These 2n rotations and reflections forms a group under the product operation of composition .This group is called the dihedral group of order 2n, and is written D_{2n}

Let A be a corner of the polygon. Write 'b' for the reflection in the line through O and A, and write 'a' for the rotation ρ_1. Then the n rotations are

$$1, a, a^2, ... , a^{n-1}$$

(where 1 denotes the identity, which leaves the polygon fixed) and the n reflections are

$$b, ab, a^2b, ... , a^{n-1}b$$

Thus all elements of D_{2n} are products of powers of a and b that is,

D_{2n} is generated by a and b.

The relations $a^n = 1$, $b^2 = 1$ and $b^{-1}ab = a^{-1}$ determine the product of any two elements of this group. We summarize all this in the presentation

$$D_{2n} = < a, b: a^n = 1, b^2 = 1, b^{-1}ab = a^{-1} >$$

3) Let n be a positive integer. The set of all permutations of $\{1, 2, 3, ..., n\}$, under composition is a group called Symmetric Group, written as S_n .the order of S_n is n!

4) Let F be either R or C. The set of all invertible $n \times n$ matrices with entries from F forms a group under matrix multiplication known as the General Linear Group of degree n over F. It is denoted as GL (n, F).It is an infinite group.The identity element of this group is the identity matrix denoted by I_n.

<u>Note</u>

A group G is said to be abelian if gh=hg for all g and h in G.

HOMOMORPHISMS

Given two groups G and H, those functions from G to H which preserve the group structure are called Homomorphism.

1.3 Definition

If G and H are groups, then a homomorphism from G to H is a function $\theta: G \to H$ such that

$$\theta(g_1 g_2) = \theta(g_1).\theta(g_2) \quad \forall\ g_1, g_2 \in G$$

<u>Note</u>

An invertible homomorphism is called an isomorphism .If there is an isomorphism from G to H then G and H are said to be isomorphic and we write $G \cong H$

1.4 Example

Let $G = D_{2n} =< a, b: a^n = 1 , b^2 = 1 , b^{-1}ab = a^{-1} >$

write the 2n elements of G in the form

$$a^i b^j ; 0 \le i \le n - 1 \ and \ 0 \le j \le 1$$

Let H be any group and suppose that H contains elements x and y which satisfy $x^n = y^2 = 1 , y^{-1}xy = x^{-1}$

Then $\theta: G \to H$ defined by $\theta(a^i b^j) = x^i y^j$ is a homomorphism.

<u>Proof</u>

Suppose that $0 \le r \le n - 1 \ \ and \ 0 \le s \le 1$

$$0 \le t \le n - 1 \ \ and \ 0 \le u \le 1$$

Then $a^r b^s a^t b^u = a^i b^j$ for some i and j with $0 \le i \le n-1,\ 0 \le j \le 1$

Moreover i and j are determined by repeatedly using the relations

$$a^n b^2 = 1,\ b^{-1}ab = a^{-1}$$

Since we have $x^n = y^2 = 1$, $y^{-1}xy = x^{-1}$ we can also deduce that

$x^r y^s x^t y^u = x^i y^j$

$\therefore \theta(a^r b^s a^t b^u) = \theta(a^i b^j) = x^i y^j = x^r y^s x^t y^u = \theta(a^r b^s).\theta(a^t b^u)$ ■

Kernels

Let G and H be groups and suppose that θ : G → H is a homomorphism.

We define the kernel of θ by Ker θ = $\{g \in G \text{ such that } \theta(g) = 1\}$ where 1 is taken as the identity element in H. Here Image of θ is denoted as Im θ.

Note:

→Ker θ is a normal subgroup of G.

→Im θ is a subgroup of H

1.5 Theorem

Suppose that G and H are groups and let θ : G → H be a homomorphism.

Then G/Ker θ $\cong$ Im θ

Group Action

1.6 Definition

Let X be a set and let G be a group. An action of G on X is a map

$* : G\times X\rightarrow X$ such that

1. ex=x

2. $(g_1 g_2)(x) = g_1(g_2 x)$ $\forall\ x \in X\ and\ \forall g_1, g_2 \in G$

Under these conditions X is said to be a G-set.

CHAPTER 2
VECTOR SPACES AND LINEAR TRANSFORMATIONS

Vector spaces

2.1 Definition

Let F be either R (the set of real numbers) or C (the set of complex numbers). A vector space over F is a set V, together with a rule for adding any two elements u, v of V to form an element u + v of V, and a rule for multiplying any element v of V by any element λ of F to form an element λv of V. (The latter rule is called scalar multiplication.) Moreover, these rules must satisfy:

(a) V is an abelian group under addition;

(b) for all u, v in V and all λ , μ in F,

(1) λ(u +v) = λu + λv,

(2) (λ + μ)v = λv + μv,

(3) (λμ)v =λ(μv),

(4) 1v =v.

The elements of V are called vectors, and those of F are called scalars.
We write 0 for the identity element of the abelian group V under addition.

2.2 Example

For each positive integer n, consider row vectors $(x_1, x_2, \ldots\ldots x_n)$ where $x_1, x_2, \ldots\ldots x_n$ belongs to F. We denote the set of all such row vectors by F^n. Define addition and scalar multiplication on F^n by

$$(x_1, x_2, \ldots\ldots x_n) + (x_1', x_2', \ldots\ldots, x_n') = (x_1 + x_1', x_2 + x_2' \ldots x_n + x_n')$$

$$\lambda(x_1, x_2, ... x_n) = (\lambda x_1, \lambda x_2, ... \lambda x_n)$$

Then F^n is a vector space over F.

Bases of vector spaces

Let $v_1, v_2, ...v_n$ be vectors in a vector space V over F. A vector v in V is a linear combination of $v_1, v_2, ...v_n$ if

v $=\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n$ for some $\lambda_1, \lambda_2, ... \lambda_n$ in F.

The vectors $v_1, v_2, ...v_n$ are said to span V if every vector in V is a linear combination of $v_1, v_2, ...v_n$.

We say that $v_1, v_2, ...v_n$ are linearly dependent if

$\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n = 0$ for some $\lambda_1, \lambda_2, ... \lambda_n$ in F not all of which are zero; otherwise $v_1, v_2, ...v_n$ are linearly independent.

2.3 Definition

The vectors $v_1, v_2, ...v_n$ form a basis of V if they span V and are linearly independent.

The number of vectors in a basis of a vector space V is called as the dimension of V and is denoted as dim V. If V = {0} then dim V = 0. The vector space V is n-dimensional if dim V = n

2.4 Example

Let V = F^n. Then

(1, 0, 0, ……… 0), (0, 1, 0, ……… , 0), ……… , (0, 0, 0, ………, 1)

is a basis of V, so dim V = n.

Another basis is

(1, 0, 0, ………, 0), (1, 1, 0, ………, 0),……… , (1, 1, 1, ……… , 1)

Linear Transformations

2.5 Definition

Let V and W be vector spaces over F. A linear transformation from V to W is a function θ: V → W which satisfies

→θ(u + v)= θ(u) +θ(v) for all u, v ∈ V , and

→θ(λv)= λ.θ(v) for all λ∈ F and v ∈ V

Endomorphisms

2.6 Definition

A linear transformation from a vector space V to itself is called an endomorphism of V.

Example: Identity Function defined on a vector space.

Matrices

Let V be a vector space over F, and let θ be an endomorphism of V. Suppose that $v_1, v_2, ...v_n$ is a basis of V and denote it as B . Then there are scalars a_{ij} in F (1 < i < n, 1 < j < n) such that for all i,

$$\theta\left(v_i\right) = a_{i1}v_1 + ... + a_{in}v_n$$

2.7 Definition

The n× n matrix (a_{ij}) is called the matrix of θ relative to the basis B, and is denoted by $[\theta]_B$

2.8 Examples

(1) If $\theta = 1_v$ (so that θ(v) = v for all v ∈ V), then $[\theta]_B = I_n$ for all bases B of V, where I_n denotes the n× n identity matrix.

(2) Let V $=R^2$ and let θ be the endomorphism (x, y)→ (x + y, x -2 y) of V. If B is the basis (1, 0), (0, 1) of V and B' is the basis (1, 0), (1, 1) of V, then

$[\theta]_B = (1\ 1\ 1\ -\ 2)$ $[\theta]_{B'} = (0\ 1\ 3\ -\ 1)$

Note

If A is an n ×n matrix over F, then the function v →vA (v ∈ F^n) is an endomorphism of F^n where A is matrix over F

2.9Definition

Let B= $\{v_1, v_2, ...v_n\}$ be a basis of the vector space V, and let

$B' = \{v'_1 v'_2 ......v'_n\}$ be a basis of V. Then for 1 ≤ i ≤ n,

$v'_i = t_{i1}v_1 + ... + t_{in}v_n$ for certain scalars t_{ij}. The n × n matrix T = (t_{ij}) is invertible, and is called the change of basis matrix from B to B'

The inverse of T is the change of basis matrix from B' to B .

2.10 Definition

If B and B' are bases of V and φ is an endomorphism of V

then $[\varphi]_B = T^{-1} [\varphi]_{B'} T$, where T is the change of basis matrix from B to B'.

CHAPTER 3

GROUP REPRESENTATION

In this chapter, we will focus on visualizing a group G as a group of matrices which we call as representation of G. We will give some examples of some representations and also introduce the concept about of representations. We will also discuss about the Kernel of a representation.

REPRESENTATIONS

Let G be a group. Let F be R or C. Remember that, GL(n,F) means the group of invertible n×n matrices with entries taken from F.

3.1 Definition

A representation of G over F is a homomorphism ρ from G to GL (n, F), for some n. The degree of ρ is the integer n. Thus if ρ is a function from G to GL(n, F), then ρ is a representation if and only if

$$\rho(gh) = \rho(g).\rho(h) \quad \forall\, g,\, h \in G$$

Since a representation is a homomorphism, it follows that for every representation ρ: G →GL (n, F), we have

$$\rho 1 = I_n \text{ and}$$

$$\rho(g^{-1}) = (\rho(g))^{-1} \qquad \forall\ g \in G$$

where I_n denotes the n×n identity matrix.

3.2 Result

Every group has representations of arbitrarily large degree.

Proof:

Let G be any group. Define ρ: G → GL (n, F) by

$$\rho(g) = I_n \quad \forall\, g \in G,$$

where I_n is the n×n identity matrix, Then

$$\rho(gh) = I_n = I_n . I_n = \rho(g).\, \rho(h) \quad \forall\, g,\ h \in G$$

so ρ is a representation of G.■

3.3 Example

Let G be the dihedral group $D_8 = \langle a, b : a^4 = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle$

Define A = $(0\ 1\ \ -\ 1\ 0\ )$ and B = $(1\ 0\ 0\ \ -\ 1\ )$ where A, B ϵ GL(2,F)

such that $A^4 = B^2 = I$, $B^{-1}AB = A^{-1}$

It follows (see Example 1.4) that the function ρ: G →GL (2, F) defined by

$$\rho\,(a^i b^j) = A^i B^j\ ;\ (0 \le i \le 3)\ (0 \le j \le 1)$$ is a homomorphism.

∴ ρ is a representation of D_8 over F. The degree of ρ is 2.

The matrices $\rho(g)$ of all elements in D_8 are given below:

$(1\ 0\ 0\ 1\ )$ $(0\ 1\ \ -\ 1\ 0\ )$ $(-\ 1\ 0\ 0\ \ -\ 1\ )$ $(0\ \ -\ 1\ 1\ 0\ )$ $(1\ 0\ 0\ \ -\ 1\ )$ $(0\ \ -\ 1\ \ -$

$g = 1$ $g = a$ $g = a^2$ $g = a^3$ $g = b$ $g = ab$

$(-\ 1\ 0\ 0\ 1\ )\quad (0\ 1\ 1\ 0\ )$

$g{=}a^2b \qquad g{=}a^3 b$

Equivalent Representations

Now we discuss a way for converting a given representation into another one.

Let ρ: G → GL (n, F) be a representation of G over F. Let T be an invertible n×n matrix over F. For all n×n matrices A and B, we have

$$(T^{-1}AT)(T^{-1}BT) = T^{-1}(AB)T$$

This observation can be used to produce a new representation σ from ρ. We define ,

$$\sigma(g) = T^{-1}(\rho(g))\,T \quad \forall\ g \in G:$$

To prove that σ is a representation of G over F

$\forall\, g\,,\ h \in G$,

$\sigma(gh) = T^{-1}(\rho(gh))\,T$ (by definition of σ)

$= T^{-1}(\rho(g)\rho(h))T$ (since ρ is a homomorphism)

$= T^{-1}(\rho(g))T.\,T^{-1}(\rho(h))T$

$= \sigma(g).\,\sigma(h)$

∴ σ is a representation.■

3.4 Definition

Let ρ: G → GL (m, F) and σ : G → GL (n, F) be representations of G over F. We say that ρ is equivalent to σ if n = m and there exists an invertible n×n matrix T such that

$$\sigma(g) = T^{-1}(\rho(g))\, T \quad \forall\ g \in G$$

<u>Note</u>

For all representations $\rho, \sigma\ and\ \tau$ of G over F, we have

→ ρ is equivalent to ρ

→ If ρ is equivalent to σ then σ is equivalent to ρ.

→ if ρ is equivalent to σ and σ is equivalent to τ , then ρ is equivalent to τ .

ie, Equivalence of representations is an equivalence relation

3.6 Examples

(1) Let $G = D_8 = \langle a, b : a^4 = b^2 = 1,\ b^{-1}ab = a^{-1}\rangle$, and consider the representation ρ of G which appears in Example 3.3 . Thus ρ(a)=A and ρ(b)=B where,

A=$(0\ 1\ \ -\ 1\ 0\ )$ and B=$(1\ 0\ 0\ \ -\ 1\ )$

Assume that F =C, and define T = $\frac{1}{\sqrt{2}}(1\ 1\ i\ \ -\ i\ )$ .Then T^{-1} = $\frac{1}{\sqrt{2}}(1\ \ -\ i\ 1\ i\ )$

T has been constructed so that T^{-1}AT is diagonal.

∴ We have, $T^{-1}AT\ =\ (i\ 0\ 0\ \ -\ i\ )\ and\ T^{-1}BT\ =\ (0\ 1\ 1\ 0\ )$

and so we obtain a representation σ of D_8 for which

σ(a) = $(i\ 0\ 0\ -\ i)$, σ(b) = $(0\ 1\ 1\ 0)$

The representations ρ and σ are equivalent.

(2) Let G = C_2,=$< a: a^2 = 1 >$ and Let , A=$(-\ 5\ 12\ -\ 2\ 5)$

Here $A^2 = I$. Hence ρ:1→I, a→A is a representation of G.

If T= $(2\ -\ 3\ 1\ -\ 1)$ then $T^{-1}AT = (1\ 0\ 0\ -\ 1)$

and so we obtain a representation σ of G for which

σ(1)=$(1\ 0\ 0\ 1)$ and σ(a)= $(1\ 0\ 0\ -\ 1)$ and σ is equivalent to ρ.

<u>Note</u>: There are two easily recognized situations where the only representation which is equivalent to ρ is ρ itself ; they are

→when the degree of ρ is 1,

→ when $\rho(g) = I_n$ *for all g in G.*

Kernels of representations

We conclude this chapter with a discussion about the kernel of a representation ρ: G →GL (n, F). By definition it consists of elements of G for which ρ (g) is the identity matrix. Thus

$$\text{Ker}(\rho) = \{g \in G: \rho(g) = I_n\}$$

Note that Ker (ρ) is a normal subgroup of G.

It can be shown that the kernel of a representation is the whole of G, which is given by the following definition.

3.8 Definition

The representation ρ: G →GL (1, F) which is defined by ρ(g)=(1) ∀ g ∈ G, is called the trivial representation of G.

In other words we can say that the trivial representation of G is the representation where every group element is mapped to the 1×1 identity matrix.

3.9 Definition

A representation ρ: G → GL (n, F) is said to be faithful if

Ker (ρ) = {1} ie, if the identity element of G is the only element g for which ρ(g) = I_n

3.10 Proposition

A representation ρ of a finite group G is faithful if and only if Im (ρ) is isomorphic to G.

Proof

We know that Ker (ρ) is a normal subgroup of G and by Theorem the factor group G/ Ker ρ is isomorphic to Im ρ. Therefore, if Ker ρ = {1} then G $\cong$ Im ρ. Conversely, if G $\cong$ Im ρ, then these two groups have same order, and so $|Ker$ ρ|= 1; that is, ρ is faithful ■

3.11 Examples

1) The representation ρ of D_8 given by

$$\rho\left(a^i b^j\right) = (0\ 1\ \ -\ 1\ 0\ )^i (1\ 0\ 0\ \ -\ 1\ )^j$$

is faithful, since the identity is the only element g which satisfies ρ(g) = I.

The group generated by the matrices $(0\ 1\ \ -\ 1\ 0\ )\ and\ (1\ 0\ 0\ \ -\ 1\ )$

is therefore isomorphic to D_8

2) Since T^{-1} AT = I_n if and only if A = I_n it follows that all representations which are equivalent to a faithful representation are faithful.

3) The trivial representation of a group G if faithful if and only if G = {1}

Note:

→Every finite group has a faithful representation.

→A representation is faithful if it is injective

CHAPTER 4

FG MODULES

This chapter gives us an introduction to FG -modules. We study this because there is a close connection between FG -modules and representation of G.

Consider a group G. Let F be R or C. Suppose that ρ:G→GL(n,F) is a representation of G. Consider a vector space V=F^n consisting of all row vectors ($\lambda_1, \lambda_2, \ldots\ldots \lambda_n$) with $\lambda_i \in F$. Define a matrix product for all v∈V and g ∈G as v.ρ(g).

Note

→The matrix product defined above is a row vector in V

→Since ρ is a homomorphism we have v.ρ(gh)=v.(ρ(g).ρ(h)) ∀ v∈V , g,h∈G

→Since ρ(1) is the identity matrix , we have v. ρ(1)=v ∀ v∈V

→By properties of matrix multiplication , we have

(λv)(ρ(g))=λ(v.ρ(g))

(u+v)(ρ(g))=u(ρ(g))+v(ρ(g)) ∀u,v ∈V, λ∈F, g∈G

4.1 Example

Let G = $D_8 = <a, b: a^4 = b^2 = 1,\ b^{-1}ab = a^{-1}>$ and let ρ: G → GL (2, F) be the representation of G over F given in Example 3.3.

Thus ρ(a)=$(0\ 1\ \ -\ 1\ 0\ )$ and ρ(b)=$(1\ 0\ 0\ \ -\ 1\ )$. If v=(λ_1, λ_2)∈F,then we have

$$v.\ \rho(a)=(-\lambda_2, \lambda_1)$$

$$v.\ \rho(b)=(\lambda_1, -\lambda_2)$$

4.2 Definition

Let V be a vector space over F and let G be a group. Then V is an FG-module if a multiplication v. g ($v \in V$, $g \in G$) is defined, satisfying the following conditions for all u, $v \in V$, $\lambda \in F$ and g, $h \in G$:

(1) $v g \in V$
(2) $v(gh) = (v g)h$
(3) $v1 = v$
(4) $(\lambda v) g = \lambda(v g)$
(5) $(u + v) g = ug + v g$

We use the letters F and G in the name `FG-module' to indicate that V is a vector space over F and that G is the group from which we are taking the elements g to form the products v g ($v \in V$).
Note that conditions (1), (4) and (5) in the definition ensure that for all $g \in G$, the function $v \to v g$ ($v \in V$) is an endomorphism of V.

4.3 Definition

Let V be an FG-module, and let B be a basis of V. For each $g \in G$, let $[g]_B$ denote the matrix of the endomorphism $v \to v g$ of V, relative to the basis B .

The connection between FG-modules and representations of G over F is revealed in the following basic result.

4.4 Theorem

(1) If ρ: G → GL(n, F) is a representation of G over F, and V =F^n then V becomes an FG-module if we define the multiplication v g by

$$v\,g = v(\rho g) \ (v \in V,\ g \in G):$$

Moreover, there is a basis B of V such that $\rho g = [g]_B$ for all g ∈ G:

(2) Assume that V is an FG-module and let B be a basis of V.
Then the function ,g → $[g]_B$ (g ∈ G) is a representation of G over F

Proof

(1)

We have already observed that for all u, v ∈ F^n, λ∈ F and g, h ∈ G, we have,

→ v(ρg) ∈ F^n
→ v(ρ(gh)) = (v(ρg))(ρh),
→ v(ρ1) = v,
→ (λv)(ρg) = λ(v(ρg)
→ (u + v)(ρg) = u(ρg) + v(ρg):

Therefore, F^n becomes an FG-module if we define v g = v(ρg) ∀ v∈ F^n , g ∈ G

Moreover, if we let B be the basis (1, 0, 0, … , 0),(0, 1, 0, … , 0), ..,(0, 0, 0, …. , 1) of F^n, then ρg = $[g]_B$ for all g ∈ G

(2) Let V be an FG-module with basis B . Since v(gh) = (v g)h for all g, h ∈G and all v in the basis B of V, it follows that

$$[gh]_B = [g]_B [h]_B$$

In particular,

$[1]_{B=}[g]_B\,[g^{-1}]_B$ for all g ∈ G. Now v1= v for all v ∈ V, so is $[1]_B$ *is* the identity matrix.

Therefore each matrix $[g]_B$ is invertible (with inverse $[g^{-1}]_B$).

We have proved that the function g → $[g]_B$ is a homomorphism from G to GL (n, F) (where n = dim V), and hence is a representation of G over F. ■

4.5 Example

Let G = D_8 = $< a, b: a^4 = b^2 = 1,\ b^{-1}ab = a^{-1} >$ and let ρ be the representation of G over F given in Example 3.3, so we have

$$\rho(a) = (0\ 1\ \ -\ 1\ 0)\ ,\quad \rho(b) = (1\ 0\ 0\ \ -\ 1)$$

Write V = F^2 . By Theorem 4.4(1), V becomes an FG-module if we define v g =v(ρg) (v ∈ V , g ∈ G).

For instance,

$$(1, 0)\ a = (1, 0)(0\ 1\ \ -\ 1\ 0) = (0, 1)$$

If v_1, v_2 is the basis (1, 0), (0, 1) of V, then we have

$$v_1 a = v_2, \qquad v_1 b = v_1,$$
$$v_2 a = -v_1, \qquad v_2 b = v_2$$

If B denotes the basis v_1, v_2, then the representation g → $[g]_B$ (g ∈G) is just the representation ρ (see Theorem 4.4(1) again).

4.6 Proposition

Assume that $v_1, v_2, \ldots v_n$ is a basis of a vector space V over F. Suppose that we have a multiplication v g for all v in V and g in G which satisfies the following conditions for all i with 1 ≤ i ≤ n, for all g, h ∈ G, and for all $\lambda_1, \lambda_2, \ldots \lambda_n \in F$:

(1) v_i g ∈V

(2) v_i (gh) = (v_ig)h

(3) $v_i 1 = v_i$

(4) $(\lambda_1 v_1 + \ldots + \lambda_n v_n)\, g = \lambda_1 (v_1 g) + \ldots + \lambda_n (v_n g)$.

Then V is an FG-module.

Proof

It is clear from (3) and (4) that v1 = v for all v ∈ V. Conditions (1) and (4) ensure that for all g in G, the function v → v g (v ∈ V) is an endomorphism of V. That is,

v g ∈ V ,

(λv) g =λ(v g),

(u + v) g = ug + v g, for all u, v ∈ V, λ ∈ F and g ∈ G.

Hence $(\lambda_1 u_1 + \ldots + \lambda_n u_n)h = \lambda_1 (u_1 h) + \ldots + \lambda_n (u_n h)$ (4.61) for all $\lambda_1, \lambda_2, \ldots \lambda_n \in F$, all $u_1, u_2, \ldots u_n \in V$ and all h ∈ G.

Now let v ∈ V and g, h ∈ G. Then $v = \lambda_1 v_1 + \ldots + \lambda_n v_n$ for some $\lambda_1, \lambda_2, \ldots \lambda_n \in F$, and

$$
\begin{aligned}
v(gh) &= \lambda_1 (v_1 (gh)) + \ldots + \lambda_n (v_n (gh)) && \text{by condition (4)} \\
&= \lambda_1 ((v_1 g)h) + \ldots + \lambda_n ((v_n g)h) && \text{by condition (2)} \\
&= (\lambda_1 (v_1 g) + \ldots + \lambda_n (v_n g))h && \text{by (4.61)} \\
&= (v g)h && \text{by condition (4)}
\end{aligned}
$$

We now have checked all the axioms which are required for V to be an FG-module.

Hence V is an FG-module ■

4.7 Definition

(1) The trivial FG-module is the 1-dimensional vector space V over F with $v g = v$ for all $v \in V$, $g \in G$

(2) An FG-module V is faithful if the identity element of G is the only element g for which $v g = v$ for all $v \in V$

For instance, the FD_8 -module which appears in Example 4.5 is faithful

FG-modules and Equivalent Representations

We conclude this chapter with a discussion about the relationship between FG-modules and equivalent representations of G over F. An FG-module gives us many representations for G, all of the form $g \rightarrow [g]_B$ ($g \in G$) for some basis B of V.

The next result shows that all these representations are equivalent to each other (see Definition 3.4) and moreover,

any two equivalent representations of G arise from some FG-module

4.8 Theorem

Suppose that V is an FG-module with basis B , and let ρ be the representation of G over F defined by $\rho: g \rightarrow [g]_B$ ($g \in G$)

(1) If B' is a basis of V, then the representation $\varphi: g \rightarrow [g]_{B'}$ ($g \in G$) of G is equivalent to ρ.

(2) If σ is a representation of G which is equivalent to ρ, then there is a basis B'' of V such that

$$\sigma: g \rightarrow [g]_{B''} \quad (g \in G):$$

Proof

(1)

Let T be the change of basis matrix from B to B' (see Definition 2.9). Then by (2.10), for all g ∈G, we have $[g]_B = T^{-1}[g]_{B'}$ T. Therefore φ is equivalent to ρ.

(2)

Suppose that ρ and σ are equivalent representations of G. Then for some invertible matrix T, we have

$$\rho g = T^{-1}(\sigma g)T \text{ for all } g \in G.$$

Let B'' be the basis of V such that the change of basis matrix from B to B'' is T. Then for all g ∈ G, $[g]_B = T^{-1}[g]_{B''}$ T, and so σ g =$[g]_{B''}$ ■

4.9 Example

Again let G = $C_3 = < a{:}\ a^3 = 1 >$. There is a representation ρ of G which is given by

ρ(1)=$(1\ 0\ 0\ 1)$ ρ(a)= $(0\ 1\ -1\ -1)$, ρ(a^2)=$(-1\ -1\ 1\ 0)$

If V is a 2-dimensional vector space over C, with basis $B = v_1, v_2$ then we can turn V into a CG-module as in Theorem 4.4(1) by defining

$$v_1 1 = v_1, \quad v_1 a = v_2, \quad v_1 a^2 = - v_1 - v_2$$

$$v_2 1 = v_2, \quad v_2 a = - v_1 - v_2, \quad v_2 a^2 = v_1:$$

Then we have

$[1]_B = (1\ 0\ 0\ 1)$ $\quad [a]_B = (0\ 1\ -1\ -1)$ $\quad [a^2]_B = (-1\ -1\ 1\ 0)$

Now let $u_1 = v_1$ and $u_2 = v_1 + v_2$. Then u_1, u_2 is another basis of V, which we call B'. Since

$$u_1 1 = u_1, \quad u_1 a = - u_1 + u_2, \quad u_1 a^2 = - u_2,$$

$$u_2 1 = u_2, \quad u_2 a = - u_1, \quad u_2 a^2 = u_1 - u_2,$$

we obtain the representation φ: g → $[g]_{B'}$ where

$[1]_{B'} = (1\ 0\ 0\ 1)$ $\quad [a]_{B'} = (-1\ 1\ -1\ 0)$ $\quad [a^2]_{B'} = (0\ -1\ 1\ -1)$

Note that if T=(1 0 1 1) then for all g in G, we have $[g]_B = T^{-1}[g]_{B'}T$

and so ρ and φ are equivalent, in agreement with Theorem 4.8(1)

CHAPTER 5

FG SUBMODULES AND IRREDUCIBILITY

We begin the study of FG-modules by introducing the basic building blocks of the theory - the irreducible FG-modules. First we require the notion of an FG-sub module of an FG-module. Throughout, G is a group and F is R or C.

FG -Submodules

5.1 Definition

Let V be an FG-module. A subset W of V is said to be an FG- submodule of V if W is a subspace and $wg \in W$ for all $w \in W$ and all $g \in G$.
Thus an FG-submodule of V is a subspace which is also an FG-module.

5.2 Example

For every FG-module V, the zero subspace $\{0\}$ and V itself, are FG-submodules of V.

Irreducible FG-modules

5.3 Definition

An FG-module V is said to be irreducible if it is non-zero and it has no FG submodules apart from $\{0\}$ and V.

If V has an FG-submodule W with W not equal to $\{0\}$ or V, then V is reducible.

Similarly, a representation ρ: G → GL (n, F) is irreducible if the corresponding FG-module F^n given by

$$v\,g = v(\rho g) \quad (v \in F^n,\ g \in G)$$

(see Theorem 4.4(1)) is irreducible; and ρ is reducible if F^n is reducible.

Suppose that V is a reducible FG-module, so that there is an FG-submodule W with 0 <dim W < dim V. Take a basis B_1 of W and extend it to a basis B of V. Then for all g in G, the matrix $[g]_B$ has the form

$$\left(X_g\ 0\ Y_g\ Z_g\right) \quad (5.31\ )$$

for some matrices X_g , Y_g and Z_g, where X_g is k × k (k =dim W).

A representation of degree n is reducible if and only if it is equivalent to a representation of the form (5.31), where X_g is k × k and

0 < k < n.

Notice that in (5.31), the functions $g \to X_g$ and $g \to Z_g$ are representations of G: to see this, let g, h ∈ G and multiply the matrices $[g]_B$ and $[h]_B$ given by (5.31).

Notice also that if V is reducible then dim V ≥ 2

5.4 Example

Let G = D_8 and let V =F^2 be the 2-dimensional FG-module described in Example 4.5(1). Thus G = <a, b>, and for all (λ, μ) ∈ V we have

$$(\lambda, \mu)\, a = (-\mu, \lambda), \quad (\lambda, \mu)\, b = (\lambda, -\mu)$$

We claim that V is an irreducible FG-module. To see this, suppose that there is an FG-submodule U which is not equal to V. Then dim U ≤ 1, so U = sp $((\alpha, \beta))$ for some $\alpha, \beta \in F$. As U is an FG-module, $(\alpha, \beta)b$ is a scalar multiple of (α, β), and hence either $\alpha = 0$ or $\beta = 0$. Since $(\alpha, \beta)a$ is also a scalar multiple of (α, β), this forces $\alpha = \beta = 0$, so U $= \{0\}$. Consequently V is irreducible, as claimed.

Chapter 6

Group Algebras

The group algebra of a finite group G is a vector space of dimension |G| which also carries extra structure involving the product operation on G. In a sense, group algebras are the source of all you need to know about representation theory. In particular, the ultimate goal of representation theory - that of understanding all the representations of finite groups would be achieved if group algebras could be fully analysed. Group algebras are therefore of great interest. After defining the group algebra of G, we shall use it to construct an important faithful representation, known as the regular representation of G

The Group Algebra of G

Let G be a finite group whose elements are $g_1, \ldots, g_n$, and let F be R or C. We define a vector space over F with $g_1, \ldots, g_n$ as a basis, and we call this vector space as FG. Take as the elements of FG as expressions of the form : $\lambda_1 g_1 + \lambda_2 g_2 + \ldots \lambda_n g_n \; (all\, \lambda_i \in F)$

The rules for addition and scalar multiplication in FG are the natural ones: namely, if

$$u=\sum_{i=0}^{n} \lambda_i g_i \quad \text{and} \quad v=\sum_{i=0}^{n} \mu_i g_i$$

are elements of FG, and λ,μ ∈ F, then

$$u + v = \sum_{i=0}^{n} (\lambda_i + \mu_i)\, g_i \quad \text{and} \quad \lambda u = \sum_{i=0}^{n} (\lambda\lambda_i)\, g_i$$

With these rules, FG is a vector space over F of dimension n, with basis $g_1, \ldots, g_n$ .The basis $g_1, \ldots, g_n$ is called the natural basis of FG.

6.1 Example

Let G = $C_3 = < a : a^3 = e >$. (To avoid confusion with the element 1 of F, we write e for the identity element of G, in this example.) The vector space CG contains

$$u = e - a + 2a^2 \quad \text{and } v = \frac{1}{2}e + 5a$$

We have

$$u + v = \frac{3}{2}e + 4a + 2a^2, \qquad \frac{1}{3}u = \frac{1}{3}e - \frac{1}{3}a + \frac{2}{3}a^2$$

Sometimes we write elements of FG in the form $\sum_{i=0}^{n} \lambda_g g \quad (\lambda_g \in F)$

Now, FG carries more structure than that of a vector space - we can use the product operation on G to define multiplication in FG as follows:

$$\left(\sum_{g\in G} \lambda_g g\right)\left(\sum_{h\in G} \mu_h h\right) = \sum_{g,h\in G} \lambda_g \mu_h (gh)$$

$$= \sum_{g\in G} \sum_{h\in H} (\lambda_h \mu_{h^{-1}g}) g$$

Where all λ_g, $\mu_h \in F$

6.2 Example

If G = C_3 and u, v are the elements of CG which appear in Example 6.1, then

$$uv = (e - a + 2a^2)(\tfrac{1}{2}e + 5a)$$

$$\tfrac{1}{2}e + 5a - \tfrac{1}{2}a - 5a^2 + a^2 + 10a^3$$

$$\tfrac{21}{2} e + \tfrac{9}{2} a - 4a^2$$

6.3 Definition

The vector space FG, with multiplication defined by

$$\left(\sum_{g\in G} \lambda_g g\right)\left(\sum_{h\in G} \mu_h h\right) = \sum_{g,h\in G} \lambda_g \mu_h (gh)$$

(λ_g, $\mu_h \in F$) is called the group algebra of G over F.

The group algebra FG contains an identity for multiplication, namely the element 1e (where 1 is the identity of F and e is the identity of G). We write this element simply as 1.

6.4 Proposition

Multiplication in FG satisfies the following properties, for all $r, s, t \in FG$ and $\lambda \in F$:

(1) $rs \in FG$

(2) $r(st) = (rs)t$

(3) $r1 = 1r = r$

(4) $(\lambda r)s = \lambda(rs) = r(\lambda s)$

(5) $(r + s)t = rt + st$

(6) $r(s + t) = rs + rt$

(7) $r0 = 0r = 0$.

In fact, any vector space equipped with a multiplication satisfying properties (1) to (7) of Proposition 6.4 is called an algebra. We shall be concerned only with group algebras, but it is worth pointing out that the axioms for an algebra mean that it is both a vector space and a ring.

The regular FG-module

We now use the group algebra to define an important FG-module.

Let $V = FG$, so that V is a vector space of dimension n over F,

where $n = |G|$. For all $u, v \in V$, $\lambda \in F$ and $g, h \in G$, we have

$$vg \in V,$$

$$v(gh) = (vg)h,$$

$$v1 = v,$$

$$(\lambda v)g = \lambda(vg),$$

$$(u + v)g = ug + vg,$$

by parts (1), (2), (3), (4) and (5) of Proposition 6.4, respectively.

Therefore V is an FG-module.

6.5 Definition

Let G be a finite group and F be R or C. The vector space FG, with the natural multiplication v g (v ∈ FG, g ∈ G), is called the regular FG-module. The representation g → $[g]_B$ obtained by taking B to be the natural basis of

FG is called the regular representation of G over F.

Note

The regular FG-module has dimension equal to |G|.

6.6 Proposition

The regular FG-module is faithful.

Proof :

Suppose that g ∈ G and v g = v for all v ∈ FG. Then 1 g = 1, so g = 1, and the result follows.

6.7 Example

Let G = $C_3 = \langle a : a^3 = e \rangle$. The elements of FG have the form

$$\lambda_1 e + \lambda_2 a + \lambda_3 a^2 \quad (\lambda_i \in F)$$

We have,

$$(\lambda_1 e + \lambda_2 a + \lambda_3 a^2)e = \lambda_1 e + \lambda_2 a + \lambda_3 a^2$$

$$(\lambda_1 e + \lambda_2 a + \lambda_3 a^2)a = \lambda_3 e + \lambda_1 a + \lambda_2 a^2,$$

$$(\lambda_1 e + \lambda_2 a + \lambda_3 a^2)\, a^2 = \lambda_2 e + \lambda_3 a + \lambda_1 a$$

By taking matrices relative to the basis e, a, a^2 of FG, we obtain the regular representation of G:

e →(1 0 0 0 1 0 0 0 1), a→(0 1 0 0 0 1 1 0 0), a^2→ (0 0 1 1 0 0 0 1 0)

FG acts on an FG-module

You will remember that an FG-module is a vector space over F, together with a multiplication vg for v 2 V and g 2 G (and the multiplication satisfies various axioms). Now, it is sometimes helpful to extend the definition of the multiplication so that we have an element vr of V for all elements r in the group algebra FG. This is done in the following natural way.

6.8 Definition

Suppose that V is an FG-module, and that v ∈ V and r ∈ FG; say

$r = \sum_{g \in G} \mu_g g \quad \mu_g \in F$ Define vr by

$$vr = \sum_{g \in G} \mu_g (vg)$$

6.9 Example

If V is the regular FG-module, then for all v ∈ V and r ∈ FG, the element vr is simply the product of v and r as elements of the group algebra, given by Definition 6.3.

CONCLUSION

A study on Groups as a matrices helps us to understand finite groups in a better way . It gives as an idea about Group Representations as a Linear Representation. Here we discussed Group representation theory in detail by using FG Modules. The topic on irreducible representation help us to study about the building blocks of representations in more detail.

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