A STUDY ON MODULE

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Submitted By

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CERTIFICATE

This is to certify that the dissertation entitled **"A STUDY ON MODULE"** submitted by SARANYA S RAJ is a record of work done by the candidate during the period of her study under my supervision and guidance.

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DECLARATION

I hereby declare that the project report entitled **"A STUDY ON MODULE"** submitted for the M.Sc. Degree is my original work done under the supervision of Ms. Alakamohan and the project has not formed the basis for the award of any academic qualification fellowship or other similar title of any other university or board.

Place: Thrikkakara

Date: Saranya S Raj

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CONTENTS

ABSTRACT

In Mathematics, a module is a generalization of the notion of vector space, where in the field of scalars is replaced by a ring. Much of the theory of modules consists of extending as many of the desirable properties of vector spaces as possible to the realm of modules over a well-behaved ring such as principal ideal domain. However, modules can be quite a bit more complicated than vector spaces. In this project, the first chapter deals with rings and vector space. The second chapter discusses about modules, sub modules and module homomorphism. In the third chapter we study about exact sequences and different types of modules. And the fourth chapter discusses about modules of finite length.

INTRODUCTION

In a vector space, the set of scalars is a field acts on the vectors by scalar multiplication, subject to certain axioms such as distributive law. In a module, the scalars need only be ring. So the module concept represent a significant generalisation. In commutative algebra, both ideals and quotient rings are modules, so that many arguments about ideals and quotient ring can be combined into a single argument about modules. In non-commutative algebra, both ideals and quotient rings are modules, so that many arguments about ideals or quotient rings can be combined into single argument about modules. In commutative algebra, the distinction between left ideals, and modules become more pronounced, though some ring-theortetic conditions can be expressed either about left ideals or left modules.

The general notion of a module was first encountered in the 1860’s till 1880’s in the work of R.Dedekind and L.Kronecker devoted to the arithmetic of algebra number and function fields. At the same time research on finite dimensional associative algebras, in particular, groups algebra of finite groups led to the study of ideals of ring which is non commutative. At first the theory of module was developed primarily as a theory of ideals of a ring. Only later it was observed that it was more convenient to formulate and prove many results in terms of arbitrary modules and not just ideals.

CHAPTER – 1

BASIC CONCEPTS

This chapter discuss some basic definitions and results that are necessary to develop this project.

1.1 DEFINITION

An algebraic structure$< G,*>$, where G is a non empty set and * is a binary operation on G , is called a group if it satisfies the following properties.

- Associative,$(a * b) * c = a * (b * c)$,for all a,b,c ϵ G
- Existence of identity. There exists an element a ϵG, there exists an identity such that a $a * e = e * a$ for all a ϵ G.
- Existence of inverse. For each element a ϵ G, there exist an element a-1 ϵ G such that $a * a - 1 = e = a - 1 * a$.

1.2 DEFINITION

A group $< G,*>$, is said to be abelian or commutative if its binary operation is commutative ie, $a * b = b * a$ for all a, b ϵ G.

1.3 DEFINITION

Let G be a group and H a subgroup of G where $H = \{h_1, h_2, \ldots.\}$. Then the set $\{h_1 a, h_2 a, \ldots..\}$ denoted by Ha is called the right cosets of H in G generated by a and the set $\{ah_1, ah_2, \ldots.\}$ denoted by aH is called the left coset of H in G generated by a.

1.4 DEFINITION

A partially ordered set is a non-empty set A together with relation R on A×A (called partially ordering of A) which is reflexive, transitive and antisymmetric.

1.5 Zorn's Lemma

If A is a nonempty partially ordered set such that every chain in A has an upper bound in A, then A contains A contains a maximal element.

1.6 DEFINITION

Let G be a group, G is said to be cyclic if there exists an ϵ G such that every element X of G, can be written in the form a^n for some n ϵ Z. The cyclic group generated by a is denoted by $< a >$.

1.7 THEOREM

Every finite cyclic group is isomorphic to the additive group Z and every finite cyclic group of order m is isomorphic to the additive group Z_m.

1.8 THEOREM

Let H be a subgroup of a group G.

- G is the union of right (resp. left) cosets of H in G.
- Two right (resp. left) cosets of H in G are either disjoint or equal.
- For every a, b ϵ G, Ha = Hb $\Leftrightarrow$ ab^{-1} ϵ H and aH = bH iff $a^{-1}b$ ϵ H.

1.9 DEFINITION

A ring is non empty set R together with two binary operator addition (+) and multiplication(.) such that

- $(R,+)$ is an abelian group
- $(R,.)$ is a semigroup
- $(ab).c = a.(bc)$ for all a, b, c ϵ R [associative law]
- $a.(b+c) = a.b + a.c$ [left distributive law] and
- (a + b).c = a. c +b. c [right distributive law]

1.10 DEFINITION

A non-zero element a in a ring R is said to be a left (resp. right) zero divisor if there exists a non-zero b ϵ R such that a. b=0 [resp. b. a = 0]. A zero divisor is an element of R which is both left and right zero divisor.

1.11 DEFINITION

A commutative ring with identity **I**R ≠ 0 and no zero divisors is called an integral domain.

1.12 DEFINITION

A ring with identity **I0** ≠ 0 in which every non zero elements has a units is called division ring.

1.13 DEFINITION

A ring with unity which is commutative in which every non zero element has a multiplicative inverse is called field.

1.14 DEFINITION

Let R be a ring and S a nonempty subset of R, that is closed under the operation addition and multiplication in R. If S is itself a ring under these operation is called a subring of R.

1.15 DEFINITION

A subring S of a ring R is called a

- Right ideal of R, if a ϵ S, r ϵ R $\Rightarrow$ ar ϵ S
- Left ideal of R, if a ϵ S, r ϵ R $\Rightarrow$ ra ϵ S
- Both sided ideal or simply an ideal, if S is a right ideal as well as left ideal

ie, if aϵ S, r ϵ R $\Rightarrow$ ar ϵ S and ra ϵ S.

1.16 DEFINITION

Let M be an R-Module and I be a two sided ideal contained in {a ϵ R/ax = 0, for all x ϵ M}, the annihilator of M. Clearly Ann(M) is an ideal of R.

CHAPTER – 2

MODULE

Here we introduce the concept of Modules,

Submodules and module homomorphism.

2.1 DEFINITIONS AND EXAMPLES

2.1.1 Left Module:

Let R be a ring. A left R-module M is an abelian group (M, +) together with a map R x M → M

(The image of (a, x) being denoted by ax is called the scalar multiplication or the structure map) such that

1. $a(x + y) = ax + ay$, ∀ a ϵ R and x, y ϵ M
2. $(a + b)x = ax + bx$, ∀ a, b ϵ R and x ϵ M
3. $(ab)x = a(bx)$, ∀ a, b ϵ R and x ϵ M

Elements of R are called scalars.

2.1.2 Unitary Module:

A left R- module is said to be unitary left R- module if 1.x = x, ∀ x ϵ M.

2.1.3 Right Module:

An abelian group (M, +) is called a right module if there is a map from M x R → M denoted by (x, a) → xa such that

1. $(x + y)a = xa + ya$ ∀, a ϵ R and x, y ϵ M
2. $x(a + b) = xa + xb$, ∀ a, b ϵ R and x ϵ M
3. $x(ab) = (xa)b$, ∀ a, b ϵ R and x ϵ M

2.2 EXAMPLES OF MODULES

1. If M is a vector space over the field R, then M is an R- module.
2. Every ring R is an R-module over itself.
3. Every additive abelian group is a module over the ring of integers.
4. If M = $M_{mn}(R)$ be the set of all $m \times n$ matrices with entries in R. Then M is an R-module, where addition is ordinary matrix addition and multiplication of each entry A by C.

2.3 DEFINITION

Let A and B be modules over a ring R. A function f: A→ B is an R- module homomorphism provided that ∀ a, c ϵ A and r ϵ R.

1. $f(a+c) = f(a) + f(c)$

2. $f(ra) = rf(a)$

2.4 DEFINITION

A homomorphism f: A→B of R -modules. A and B is called

- A monomorphism if f is injective
- A epimorphism if f is surjective
- An isomorphism if f is bijective

2.5 PROPOSITION

For an abelian group M, let endz(M) be the ring of all endomorphism of M. Let R be any ring. Then we have the following

1. M is a left R module iff there exists a homomorphism of ring $\psi: R \to Endz(M)$.
2. M is R unitary iff $\psi(1R) = idM$.
3. M is a right R module iff there exists an anti-homomorphism of rings $\psi': R \to Endz(M)$.

2.6 DEFINITION

Let R be a ring. Let A be an R- module. A non-empty subset B of A is called an R-submodule of A if

1. B is an additive subgroup of A. ie, x, y ϵ B→ x-y ϵ B.
2. B is closed for scalar multiplication. ie, x ϵ B, a ϵ R→ xa ϵ B.

2.7 DEFINITION

A subset S = {a_i: $i\epsilon$I} of a module M spans if every element of M can be written as a finite sum $\sum_i (r_i, a_i)$. M is a finitely generated, written as fg, if M has a finite spanning set.

2.8 PROPOSITON

The submodule S generated by N and K is the sub module $N + K = \{x + y | x\epsilon N, y\epsilon K\}$

Proof

Clearly, $N + K$ is a submodule of M, $N \subset N + K$ and

$K \subset N + K$

So that $S \subset N + K$.

Conversely for any x϶ N, $y \epsilon K$ we have $x, y \epsilon S$

So that $x + y \epsilon S$

Thus $N + K \epsilon S$ and $S = N + K$.

2.9 DEFINITION

If x ϵ M, then $Rx = \{rx ; r \epsilon R\}$ is called the cyclic subgroup generated by X. M is cyclic module if M = Rx, for some x in M.

2.10 THEOREM

Let B be a subgroup of a module A over a ring R then the quotient group A/B is an R-module with the action of R on A/B given by $r(a + B) = ra + B$ for all $r\epsilon R, a\epsilon A$. The map

$\pi: A \rightarrow A/B$ given $by\ a \rightarrow a + b$ is an R-module epimorphism with kernel B.

Proof

Since a is an additive abelian group, B is normal subgroup and A/B is well defined abelian group.

If $a + B = a' + B$, then $a - a' \epsilon B$, since B is a submodule

$ra - ra' = r(a - a')$ for all $a \epsilon B, r \epsilon R$. Thus $ra + B = ra' + B$, because two left cosets of B in A are same and the action of R on A/B is well defined.

Consider the map $\pi: A \rightarrow A/B$ given by $a \rightarrow a + B$

First, we shall prove that the mapping π is well defined.

ie., if $a, b \epsilon A$ and $a + B = b + B$ then $\pi(a) = \pi(b)$

We have,

$a + B = b + B \Rightarrow a - b\epsilon B$

$\Rightarrow\ \pi(a-b) = e$, the identity of A/B

$\Rightarrow\ a-b+B = e$

$\Rightarrow\ a+B = b+e+B$

$\Rightarrow\ a+B = b+B$

$\Rightarrow\ \pi(a) = \pi(b)$

π is well defined.

Let $a+B$ be any element of A/B

Now a ϵ A and we have $\pi(a) = a+B$

Therefore, π is onto A/B

For $a, c\ \epsilon\ A$ and $r\ \epsilon\ R$

$\pi(a+c) = a+c+B$

$= a+B+c+B$

$\pi(a+c) = \pi(a) + \pi(c)$

$\pi(ra) = ra + B$

$= r(a+B)$

$\pi(ra) = r\,\pi(a)$.

Thus, π is an R-module epimorphism with kernel B.

2.11 DEFINITION

An R module A is said to be the direct sum of two of its submodule A_1 and A_2 written as $A = A_1 \oplus A_2$. If each element of A is uniquely expressed as sum of an element A_1 and an element of A_2.

2.12 PROPOSITION

Suppose M and N are sub module of a module P over R then $M \cap N = 0$ iff every element $Z\ \epsilon\ M+N$ can be uniquely written as $Z = x+y$ with $x\ \epsilon\ M$ and $y\ \epsilon\ N$.

Proof

Suppose $M \cap N = 0$

Say $Z = x + y = x' + y'$; $x, x'\ \epsilon\ M$ and $y, y'\ \epsilon\ N$

Then $x - x' = y' - y\ \epsilon\ M \cap N = 0$

$\Rightarrow\ x = x'$ and $y = y'$

Hence it is unique.

Conversely, suppose that every element of M +N has a unique decomposition.

Let $Z \epsilon M \cap N$

Now $0 = z + (-z) = 0 + 0 \epsilon M + N , z \epsilon M , -z \epsilon N$

By uniqueness we get z = 0

ie., $M \cap N = 0$.

2.13 REMARK

If M = Ra and N = Rb are cyclic, then $M \oplus N$ is generated by the two elements $(a, 0)$ and $(0, b)$. In general, if M_i is generated by N_J element for $1 \leq i \leq 2$, then $M1 \oplus M2$ is generated by N_1 +N_2 element.

2.14 PROPOSITION

An R- module $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ iff

1. $M = M_1 + M_2 + \cdots + M_n$
2. $Min \, (M_1 + M_2 + \cdots M_{i-1} + \cdots + M_n) = 0$

Proof

Suppose $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$

Then clearly (1) is true.

To prove (2)

Suppose x is in the intersection on LHS

So that $x \epsilon Mi$ and $x = y_1 + y_2 + \cdots + y_{i-1} + y_i + \cdots + y_n; yj \epsilon Mj, j \neq 1$

Since $x = 0 + 0 + \cdots . +0 + x + 0 + \cdots . . +0$, with x in the i th place.

By uniqueness we have x = 0

Conversely assume condition (1) and (2)

By (1), each x ϵM can be expressed as

$X = x_1 + x_2 + \cdots + x_n, x_i \epsilon M_i$

Suppose $x = y_1 + y_2 + \cdots + y_{i-1} + y_i + \cdots + y_n; yj \epsilon Mj$

Then $0 = (x1 - y1) + (x2 - y2) + \cdots . . +(xi - yi) + \cdots . . +(xn - yn)$

So that $xi - yi \epsilon Mi$ and

$xi - yi = [(x1 - y1) + (x2 - y2) + \cdots . + (xi - 1 - yi - 1) + (xi + 1 - yi + 1) \ldots +$
$(xn - yn)] \epsilon M_1 + M_2 + \cdots M_{i-1} + M_i + \cdots + M_n$

Hence by (2), $xi - yi = 0$ $ie, xi = yi$; $1 \leq i \leq n$

Thus $M = M1 \oplus M2 \oplus \ldots\ldots \oplus Mn$.

2.15 REMARK

$\prod_{i \epsilon I} \boldsymbol{A_i}$ is called the direct product of the family of R – modules $\{A_i / i\epsilon I\}$.

2.16 THEOREM

If R is a ring $\{Ai \mid i\epsilon I\}$, a family of R module, C an R –module and $\{\psi i : C \rightarrow Ai \mid i\epsilon I\}$ a family of R- module homomorphism, then there is a unique R Module homomorphism $\psi i : C \rightarrow \prod_{i\epsilon I} \boldsymbol{A_i}$ such that $\Pi i \; \psi = \psi 1 \; for \; all \; i \; \epsilon \; \boldsymbol{I}$. $\prod_{i\epsilon I} \boldsymbol{A_i}$is uniquely determined upto isomorphism by this property. In otherwords $\prod_{i\epsilon I} \boldsymbol{A_i}$ is product in the category of R-modules.

Proof

Let $\{Ai / i\epsilon I\}$ be a family of R –module and $\{ \psi i : C \rightarrow Ai \mid i\epsilon I\}$ a family of R- module homomorphism, then by theorem there is a unique group homomorphism $\psi i : C \rightarrow \Pi i \; \epsilon \; I \; Ai$ which has the desired property given by $\psi(c) = \{\psi_i(c)\}$.

Since each ψ_i is an R- module homomorphism.

$\psi(rc) = \{ \psi_i(rc)\} = \{r\psi_i(c) \} = r\{\psi_i(c)\}$

$\therefore \psi(rc) = r\,\psi(c)$.

And ψ is an R – module homomorphism.

Thus $\prod_{i\epsilon I} \boldsymbol{A_i}$ is a product in the category of R-module and therefore determined upto isomorphism.

CHAPTER – 3

TYPES OF MODULES

This chapter discuss about different types of modules

3.1 DEFINITION

An R- Module A is called a free module if A has a basis B. That is, linearly independent subset B of A such that A is spanned by B over R.

3.1.1 REMARK

An R- module-Rx is called free if $ann\ (x)\ = 0$. An R module M is called free if it can be expressed as a direct sum M ⊕Σ Mα, where each Mα is a free cyclic R – module α.

3.1.2 EXAMPLE

For any ring R with unity, the left R- module is free with basis {1} or {u}, u is any unit in R.

3.2 THEOREM

A Vector space is a free module.

Proof

Let V be a vector space over a division ring D. Let $\mathcal{F}$ be the family of all linearly independent subjects of V.

Let $\mathcal{F}\ =\ \{A\ \subseteq\ V/\text{A is linearly independent over D}\}$ We can see that $\mathcal{F}\ \neq\ \phi$, because $\mathcal{F}$ contain all non-zero elements of V.

Partially order $\mathcal{F}$ under set inclusion and apply zorn's lemma to get a maximal element B in $\mathcal{F}$.

Claim: B is a basis for V.

We have to show that B spans V. If not, then there exists a v ϵ V such that v is not a linear combination of any finite subset of B.

Now $B\text{‘}\ =\ B\ \cup\ \{V\}$ is a linearly independent, because

$$\alpha v\ +\ \sum_{i=1}^{r} \alpha_i b_i =\ 0$$

$\Rightarrow\ \alpha\ \neq\ 0$

$\Rightarrow v\ =\ -\alpha\ [\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_r b_r]$

$=\ -\alpha^{-1}\alpha_1 b_1 - \alpha^{-1}\alpha_2 b_2 - \cdots - \alpha^{-1}\alpha_r b_r$

$\Rightarrow\ V\ \epsilon\ Span\ (B)$, which is a contradiction.

Thus $B' \epsilon \mathcal{F}$ which contradicts the maximality of B.

Hence B spans V over D.

3.3 PROPOSITION

Suppose $\mathcal{F} : m \rightarrow N$ is an onto module map, and N and $K = Ker\ f$ are free of respective rank m and n. then m is free of respective rank m +n.

Proof

Let $\{ \mathcal{F}\ (b_1), \ldots\ , \mathcal{F}\ (b_m)\}$ and $B = \{ b_1', \ldots, b_n' \}$be the bases of N and K respectively.

We have to show that $B = \{b_1, \ldots, b_m, b_1\}$ is a base of M.

First, we have to show that B is a spanning set, for any a in M, we have r_i in R such that

$\mathcal{F}(a) = \Sigma\, r_i \mathcal{F}(b_i) = \mathcal{F}(\Sigma r_i b_i)$

$So\ that\ \ \mathcal{F}\ (a - \Sigma r_i b_i) = 0$

$\Rightarrow\ a - \Sigma r_i b_i \ \epsilon\ K$

and thus has the form $\Sigma r_i' b_i'$

Hence $a = \Sigma r_i b_i + \Sigma r_i' b_i' = 0$

Thus 0 $= \mathcal{F}\ (\Sigma r_i b_i) + \mathcal{F}(\Sigma r_i' b_i') = \mathcal{F}(\Sigma\, r_i b_i) + 0 = \Sigma\, r_i \mathcal{F}(b_i)$

Implying each $r_i = 0$

Hence the theorem.

3.4 DEFINITION

A module P is projective if and only if it satisfies one of the following equivalent conditions:

- for every surjective module homomorphism $f: N \rightarrow M$ and every module homomorphism $g: P \rightarrow M$, there exists a module homomorphism $h: P \rightarrow N$ such that $fh = g$.
- Every short exact sequence of modules of the form $0 \rightarrow A \rightarrow B \rightarrow P \rightarrow 0$ is a spit exact sequence.

3.4.1 EXAMPLES

- Direct sums and direct summands of projective modules are projective.

- If $e = e^2$ is an idempotent in the ring R, then Re is a projective left module over R.

3.4.2 REMARKS

Any free module is projective. The converse is true in the following cases;

- If R is a field or skew field: any module is free in this case.
- If the ring R is a principal ideal domain. Foe example, this applies to R = Z, so an abelian group is projective if and only if it is a free abelian group.
- If the ring R is a local ring.

3.5 DEFINITION

A left module Q over the ring R is injective if and only if it satisfies one of the following equivalent conditions:

- If Q is a submodule of some other left R-module M, then there exists another K of M such that M is the internal direct sum of Q and K, i.e., $Q + K = M \; and \; Q \cap K = \{0\}$.
- Any short exact sequence $0 \to Q \to M \to K \to 0$ of left R-modules split.
- If X and Y are left R-modules, $f: X \to Y$ is an injective module homomorphism and $g: X \to Q$ is an arbitrary module homomorphism, then there exists a module homomorphism $h: Y \to Q$ such that $hf = g$.

3.5.1 EXAMPLES

- Trivially, the zero module $\{0\}$ is injective.
- Given a field K, every K-vector space Q is an injective K-module.

3.6 DEFINITION

A module M over a ring R is flat if the following condition is satisfied:

For every injective linear map $\varphi: K \to L$ of R-modules, the map

$\varphi \otimes_R M : K \otimes_R M \to L \otimes_R M$ is also injective where $\varphi \otimes_R M$ is the map induced by $k \otimes m \to \varphi(k) \otimes m$.

3.6.1 REMARK

Every projective module is flat. The converse is in general not true: the abelian group Q is a Z-module which is flat, but not projective.

Conversely, a finitely related flat module is projective.

In general, the precise relation between flatness and projectivity shows that a module M is projective if and only if it satisfies the following conditions:

- M is flat
- M is a direct sum of countably generated modules,
- M satisfies a certain Mittag-Leffler type condition.

3.7 DEFINITON

A module M over a ring R is called torsionless if it can be embedded into some direct product R^l. i.e., M is torsionless if each non-zero element of M has non-zero image under some R- linear functional $f: f \in M^* = Hom_R(M, R), f(m) \neq 0$.

3.7.1 EXAMPLES

- A unital free module is torsionless. More generally, a direct sum of torsionless modules is torsionless.
- A submodule of a torsionless module is torsionless. In particular, any projective module over R is torsionless.
- If R is a commutative ring which is an integral domain and M is a finitely generated torsion-free module then M can be embedded into R^n and hence M is torsionless.

3.8 DEFINITION

A torsion-free module is a module over a ring such that zero is the only element annihilated by a regular element (non-zero divisor) of the ring. In other words, a module is torsion free if its torsion submodule is reduced to its zero element.

3.9 ARTINIAN MODULES

3.9.1 THEOREM

The following are equivalent for an R- module M.

1. Descending chain condition (d.c.c) hols for sub modules of M. ie, any descending chain $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n$ of submodules of M is stationary in the sense that $M_r = M_{r+1} = \cdots for\ some\ r.$
2. Minimum condition for submodule hols for M, in the sense that any non-empty family of submodules of M has a minimal element.

Proof

(1) ⟹ (2)

Let = $\{M_i\ ;\ i\epsilon\ I\}$ be a non-empty family of submodule M.

Pick any $i_1\ \epsilon\ I$ and look at $M_{i1}$is minimal in $\mathcal{F}$, we are through.

Otherwise, there is an i_2 ϵ I such that

$M_{i1} \supseteq M_{i2}$, $M_{i1} \neq M_{i2}$.

If thus M_{i2} is minimal in $\mathcal{F}$, we through again.

Proceeding thus, if we do not find the minimal element at any stage, we do not find the minimal element at any finite stage, we would end up with a non-stationary descending chain of submodules M_1 namely $M_{i1} \supseteq M_{i2} \ldots M_{in} \supseteq \ldots$ contradicting (1).

(2) ⟹ (1)

Let $M_1 \supseteq M_2 \supseteq \ldots\ldots. \supseteq M_n \supseteq \ldots\ldots.$ be a descending chain of submodules of M. Consider the non-empty family

$\mathcal{F} = \{M_i\ ;\ i\epsilon\ I\}$ of submodules of M.

This must have a minimal element, say M_r, for some r.

Now we must have $M_s \subseteq M_r$, $\forall\ s \geq r$, which implies by minimality of Mr.

$ie\ M_s = M_r, \forall\ s \geq r.$

3.9.2 DEFINITION

A module M is called Artinian if d.c.c. holds for M.

3.9.3 EXAMPLE OF ARTINIAN MODULE

1. A module which has only finitely many submodules is Artinian.
2. Finite dimensional vector space is Artinian.
3. Infinite cyclic groups ae not Artinian.

3.9.4 THEOREM

1. Submodules and quotient modules of Artinian modules are Artinian.
2. If a module M is such that it has a submodule N with both N and M / N are Artinian, then M is Artinian.

Proof

(1) Let M be Artinian and N a submodule of M.

Any family of submodules of N is also one in M and hence the result follows.

On the other hand, any descending chain of submodules of M/N corresponds to one in M (wherein each member contains N) and hence the result.

(2) Let $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots \supseteq \cdots$ be a descending chain in M.

Intersecting with N gives the descending chain in N, namely,

$N \cap M_1 \supseteq N \cap M_2 \supseteq \cdots \supseteq N \cap M_n \supseteq \cdots \supseteq \cdots$ which must be stationary, say

$N \cap M = N \cap M_{r+1}$ for some T.

On the other hand, we have the descending chain in M/N, namely,

$(N + M_1)/N \supseteq (N + M_2)/N \supseteq \cdots \supseteq (N + M_n)/N \supseteq \cdots \supseteq \cdots$ which must be also stationary, say $(N + M_s)/N = (N + M_{s+1})/N = \cdots$ for some s.

Now we prove the following.

Claim: $M_n = M_{n+1}\ \forall\ n \geq (r + s)$.

This is an immediate consequence of the four facts, namely,

1. $M_n \supseteq M_{n+1}$, $\forall$ n ϵ N,
2. N ∩ M_n = N ∩ M_{n+1} $\forall$ n ≥ r,
3. (N + M_n)/N = (N + M_{n+1})/N, $\forall$ n ≥ s and
4. (N + M_n)/N ≅ M_n / (N ∩ M_n), $\forall$ n ϵ N.

Putting together we get that,

M_n / (N∩ M_n) = (N + M_n)/N = (N + M_{n+1})/N = M_{n+1}/ (N ∩ M_{n+1}) which implies the claim and hence the result.

3.9.5 COROLLRY

Every non-zero submodule of an Artinian module contains a minimal submodule.

3.9.6 REMARK

1. Direct sum of an infinite family of non-zero Artinian modules is not Artinian (because it contains non-stationary descending chains).
2. However, a sum of an infinite family of distinct Artinian modules could be Artinian.

3.10 NOETHERIAN MODULES

3.10.1 THEOREM

The following are equivalent for an R-module M.

1. Ascending chain condition (a.c.c) holds for submodules of M, i.e., any ascending chain $M_1 \subseteq M_2 \subseteq \dots \subseteq M_n \subseteq \dots \subseteq \dots$. of submodules of M is stationary in the sense that $M_r = M_{r+1} = \cdot\cdot\cdot = \cdot\cdot\cdot$ for some r.
2. Maximum condition holds for M in the sense that any non-empty family of submodules of M has a maximal element.
3. Finiteness condition holds for M in the sense that every submodule of M is finitely generated.

Proof

(1) ⟹ (2)

Let F = {M_i, iϵ I} be a non-empty family of submodules of M.

Pick any index i_1 ϵ I and look at M_{i1}.

If M_{i1} is maximal in F, we are done.

Otherwise, there is an i_2 ϵ I such that M_{i1} c M_{i2}, $M_{i1} \neq M_{i2}$.

If this M_{i2} is maximal in F, we are done again.

Proceeding thus, if we do not find a maximal element at any finite stage, we would end up with a non-stationary ascending chain of submodules of M, namely,

M_{i1} C M_{i2} C · · · C M_{in} C · · · · · · contradicting (1).

(2) ⟹ (3)

Let N be a submodule o f M.

Consider the family F of all finitely generated submodules of N.

This family is non-empty since the submodule (0) is a member.

So this family has a maximal member, say $N_0 = (x_1, x_2, \dots, x_r)$.

If $N_0 \neq N$, pick an x ϵ N, x$\not\epsilon$ N_0.

Now $N1 = N_0 + (x) = (x, x_1, x_2, \dots, x_r)$ is a finitely generated submodule of N and hence N_1 ϵ F.

But then this contradicts the maximality of N_0 in F since N_0 c N_1, $N_0 = N_1$ and so $N_0 = N$ is finitely generated.

(3) ⟹ (1)

Let $M_1 \subseteq M_2 \subseteq \dots \subseteq M_n \subseteq \dots \subseteq \dots$. be an ascending chain of submodules of M.

Consider the submodule $N = \cup_{i=1}^{\infty} M_i$ of M which must be finitely generated, say

$N = (x_1, x_2, \dots, x_n)$.

It follows that xi ϵ M_r, ∀ i, $1 \leq i \leq n$ for some r (> 0).

Now we have $N \subseteq M_s \subseteq N$ ∀ s≥ r and so $N = M_r = M_{r+1} = \dots$.

3.10.2 DEFINITION

A module M is called Noetherian if a.c.c. holds for M.

3.10.3 EXAMPLES

1. A module which has only finitely many submodules is Noetherian.
2. Finite dimensional vectorspace are Noetherian.
3. Infinite cyclic groups are Noetherian because every subgroup of a cyclcc group is again cyclic.

3.10.4 THEOREM

a) Submodules and quotient modules of Noetherian modules are Noetherian.
b) If a module M is such that it has a submodule N with both N and M/N are Noetherian, then M is Noetherian.

Proof:

a) Let M be Noetherian and N be a submodule of M.

Any family of submodules of N is also one in M and hence the result follows.

On the other hand, any ascending chain of submodules of M/N corresponds to one in M and hence the result.

b) Let $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots \subseteq \cdots$ be an ascending chain in M.

Intersecting with N gives the ascending chain in N, $N \cap M_1 \subseteq N \cap M_2 \subseteq N \cap M_n \subseteq \cdots \subseteq \cdots$ which must be stationary, say N$\cap$ $M_r = N \cap M_{r+1} = \cdots =$ for some r.

On the other hand, we have the ascending chain in M/N,
$(N + M_1)/N \subseteq (N + M_2)/N \subseteq \ldots \subseteq (N + M_n)/N \subseteq \ldots \subseteq \cdots$ which must be

also stationary, say, $(N + M_s)/N = (N + M_{s+1})/N = \ldots = \cdots =$ for some s.
This is an immediate consequence of the four facts,

1. $M_n \subseteq M_{n+1}$, $\forall$ nϵN

2. $N \cap M_n = N \cap M_{n+1}$, $\forall$ n $\geq$ r ,
3. $(N + M_n)/N = (N + M_{n+1})/N$, $\forall$ n $\geq$ s and

4. $(N + M_n)/N \simeq M_n/(N \cap M_n)$, ∀ n∈N.

Putting together we get that,

$M_n/(N \cap M_n) = (N + M_n)/N = (N + M_{n+1})/N = M_{n+1}$ / $(N \cap M_{n+1)}$ which implies the claim.

Hence the result.

3.10.5 COROLLARY

Every non-zero submodule of a Noetherian module is contained in a maximal submodule.

3.10.6 REMARK

Maximal submodules exist in a non-zero Noetherian module (because a maximal submodule is simply a maximal element I the family of all submodules N of M, $N \neq M$.

3.11 DEFINITION

A pair of module homomorphisms, A $\rightarrow^{f}$ B $\rightarrow^{g}$ C, is said to be exact at b provided $Imf = kerg$.
A finite sequence of module homomorphisms,
$A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow\rightarrow A_{n-1} \rightarrow A_n$ is exact provided $Im\ f_i =$ ker f_{i+1}, for I = 1,2,..,n-1.
An infinite sequence of module homomorphism, ……$A_{i-1} \rightarrow A_i \rightarrow A_{i+1} \rightarrow \cdots$ Is exact provided $Im\ f_i =$ ker $f_{i+1}$for all $i \epsilon Z$.

3.11.1 REMARK

0 → A → B is an exact sequence of module homomorphism iff f is a module monomorphism. Similarly, B → C → 0 is exact iff g is a module epimorphisms.

An exact sequence of the form, 0 → A → B→ C → 0 is called a short, exact sequence, f is a monomorphism and g is an epimorphism.

3.11.2 EXAMPLES

- For every submodule N of M,

$$0 \to \mathrm{N} \overset{j}{\to} M \overset{\pi}{\to} M/N \to 0,$$

  where j and π are the natural injection and projection respectively.

- For any two R-modules M and N,

$$0 \to M \overset{j_M}{\to} M \oplus N \overset{\pi_N}{\to} N \to 0,$$

  where $j_M(x) = (x, 0)$ and $\pi_N(x, y) = y$. (**Split short exact sequence).**

- Any R-linear map $\varphi: M \to N$ indues a short exact sequence

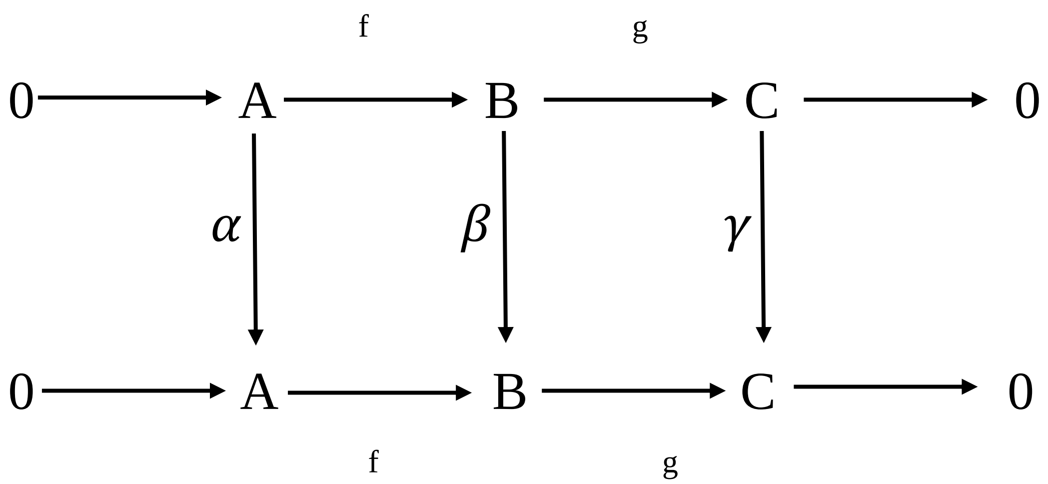
$$0 \to Ker(\varphi) \overset{j}{\to} M \overset{\varphi}{\to} Image(\varphi) \to 0,$$

  where j is the inclusion map, and φ is same as the given map but its codomain is changed to Image (φ).

3.12 LEMMA (THE SHORT FIVE LEMMA)

A commutative diagram of R- modules and R module homomorphism such that each Row is a short exact sequence. Then

- α, γ monomorphisms ⇒ β is a monomorphism
- α, γ epimorphism ⇒ β is a epimorphism
- α, γ isomorphism ⇒ β is a isomorphism



Proof

- Let $b\epsilon B$ and suppose that $\beta(b) = 0$.

We must show that b = 0.

By commutativity, we have, $\gamma g(b) = g'(b) = g'(0) = 0$

This implies g(b) = 0, since γ is a monomorphism.

By exactness of the top row at B, we have $b \epsilon \ker g = Im\, f$, say b = f(a), a ϵA.

By commutativity $f'(\alpha(a)) = \beta f(a) = \beta(b) = 0$, by exactness of the bottom row t A, f' is a monomorphism and hence $\alpha(a) = 0$.

But α is a monomorphism.

Therefore, a = 0 and hence b = f(a) = f(0) = = 0.

Then β is a monomorphism.

- Let $b'\epsilon B'$.

Then $g'(b)\epsilon C'$, since γ is an epimorphism $g'(b) =$ γ is an epimorphism $g'(b') = \gamma(c)$ for some $c\epsilon C$.

By exactness of the top row at C, g is an epimorphism, hence c = g(b) for some b ϵB.

By commutativity, $g'(\beta(b)) = \gamma g(b) = \gamma(c) = g'(b')$

Thus $g'(\beta(b) - b') = 0$ and $\beta(b) = b'\epsilon \ker g = Im\, f$ by exactness say,

$f'(a') = \beta(b) - b', a'\epsilon A'$.

Since α is an epimorphism, $a' = \alpha(a)$ for some $a\epsilon$A.

Consider b = f(a) ϵB, $\beta(b - f(b)) = \beta(b) - \beta(f(a))$

By commutativity, $\beta(f'(a)) = f'(a') = \beta(b) - b$

Hence $\beta(b - f(b)) = \beta(b) - \beta(f(a)) = \beta(b) - \beta(b - b') = b'$ and β is an epimorphism.

- α is an epimorphism, implies α is one-one and onto.

Hence α is a monomorphism and epimorphism.

Similarly, γ is a monomorphism and epimorphism.

By first and second part of proof, we have α, γ is an epimorphism implies β is an epimorphism.

β is a monomorphism and epimorphism implies β is an isomorphism.

CHAPTER – 4

MODULES OF FINITE LENGTH

4.1 DEFINITION

A module M is called simple if

- $M \neq (0)$
- M has no submodules over than (0) and M

4.1.1 EXAMPLES

- Any one-dimensional vector space is simple.
- Any minimal submodule of a module is simple.
- A submodule N of M is maximal in M⇔ M/N is simple.

4.1.3 REMARK

1. A non-zero module is simple.
2. (0) is a maximal submodule of M.
3. M is a minimal submodule of M.
4. Every non-zero element of M generates M.
5. These four conditions are equivalent. i.e., $1) \Leftrightarrow 2) \Leftrightarrow 3) \Leftrightarrow 4)$

4.1.4 REMARK

Simple submodules exist in a non-zero Artinian module while simple quotients exist for a non-zero Noetherian one.

Proof

Since minimal submodule exist in a non-zero Artinian module and minimal submodule of a module is simple.

Hence simple module exists in a non-zero Artinian module.

Since maximal submodules exist in a non-zero Noetherian module and the quotient module by maximal submodule, it becomes simple module.

Hence simple quotients exist for a non-zero Noetherian module.

4.2 DEFINITION

By a composition series of a non-zero module M, we mean a finite descending chain of submodules of M starting with M and ending with (0), say

$M = M_0 \supset M_1 \supset \cdots \supset M_m = (0)$ such that the successive quotients M_i/M_{i+1} are simple $\forall$ i. The integer m is

called length of the series. It is also called a Jordan-Holder filtration or simply a filtration.

4.2.1 EXAMPLE

- A vector space V having a finite basis $\{V_1, V_2, \dots, V_m\}$ has a composition series of length m, say, $V = V_0 \supset V_1 \supset \cdots V_m = (0)$ where V_i span of $\{V_{i+1}, V_{i+2}, \dots, V_m\}$ for all i, $0 \leq i \leq m$ with $V_m = (0)$.
- An infinite cycle group cannot have a composition series since it has no minimal submodules as if it has then it will be cyclic subgroup generated $< a^i >$ then $< a^{ik} >$ will be subgroup of $< a^i >$.

4.3 DEFINITION

A module is called a module of finite length if it is either zero or has composition series.

4.3.1 THEOREM

A module is of finite length iff it is both Artinian and Noetherian.

Proof

Let M be a module of finite length.

Case 1: If M = (0), the result is obvious.

Case 2: Suppose M ≠ (0).

Then it has a composition series, say $M = M_0 \supset M_1 \supset \cdots \supset M_m = (0)$

Now we will prove by induction on m.

If m = 1, then M is simple and hence trivially M is both Artinian and Noetherian.

Assume that m ≥ 2 and any module having some composition series of length atmost m-1 is both Artinian and Noetherian.

Now look at M_1 which has the composition series, say $M_1 \supset \cdots \supset M_m = (0)$ of length m-1.

Hence M_1 is both Artinian and Noetherian.

On the other hand, the quotient M/M_1, being simple is both Artinian and Noetherian.

[Since by a result, If M is a module such that it has a submodule N with both N and M/N are Artinian (Noetherian), then M is Artinian (Noetherian)]

Hence M is both Artinian and Noetherian.

Conversely, suppose that M is both Artinian and Noetherian.

Assume that $M \neq (0)$.

Since M is Noetherian, it has a maximal submodule, say M_1.

If $M_1 = 0$, then M is simple and hence it is a module of finite length.

If $M \neq 0$ and M_1 being Noetherian, has a maximal submodule, say M_2.

If $M_2 = (0)$, we have a composition series of M, namely $M = M_0 \supset M_1 \supset M_2 = (0)$.

At any final stage n, if $M_n \neq (0)$, we get a maximal submodule M_{n+1} of M_n and so on, yielding an infinitely descending chin of submodules of M, namely

$M = M_0 \supset M_1 \supset \cdots \supset M_n \supset \cdots$, contradicting that M is Artinian.

Hence $M_m = (0)$, for some m, as required.

4.3.2 THEOREM

Submodules and quotient modules of a module of finite length are modules of finite length.

Proof

Part 1: Let M be R-module of finite having a composition series,

$M = M_0 \supset M_1 \supset \cdots \supset M_m = (0)$ $\longrightarrow$ (1)

Let N be a submodule of M.

Intersecting (1) with N, f is R linear onto map. Observe that $ker f = (N + M_{i+1}) / N$.

Hence $(N + M_i) / N / (N + M_{i+1}) / N \cong M_i / M_{i+1}$ is simple.

Deleting repetitions, we get composition series for M/N of the form,

$M/N = \eta(M_0) \supset \eta(M_{j1}) \supset \cdots \supset \eta(M_{js}) = (0)$.

Hence M/N is of finite length.

4.3.3 THEOREM

If N is of the finite length and M/N is of finite length, then M is of finite length.

Proof

Let N be a submodule of such that N and M/N are of finite length, with composition series of lengths n and m respectively,

i.e., $N = N_0 \supset N_1 \supset N_n = (0)$

$M/N = (M/N)_0 \supset (M/N)_1 \supset \cdots (M/N)_M = (0)$.

For $0 \le i \le m$, let $M_i = \eta^{-1}((M/N)_i)$, i.e., we have

$$M = M_0 \supset M_1 \supset \cdots M_m = N$$

For $0 \le i \le m-1, M_i \,/\, M_{i+1} \cong ((M/N)_i/(M/N)_{i+1})$ is simple.

Now we have a composition series for M which is of length n + m,

$M = M_0 \supset M_1 \supset \cdots M_m = N = N_0 \supset N_1 \supset \cdots N_n = (0)$.

If the composition series of N and M/N contains n and m terms respectively.

Then composition series of M contains m + n terms.

4.4 THEOREM (JORDAN – HOLDER)

Any two composition series of a non-zero module are equivalent.

i.e., Let $M = M_0 \supset M_1 \supset \cdots M_m = (0)$ and $M = N_0 \supset N_1 \supset \cdots N_n = (0)$ be any two composition series of M. Then, M = n and

$\forall\, i, 0 \le i \le m-1, \exists\, j = j(1), 0 \le j \le n-1$ such that $M_i/M_{i+1} \cong N_j/N_{j+1}$ and vice versa.

Proof

We will prove the result by induction on the length of one of the composition series, say m.

- If m = 1, then M is simple and hence $N_1 = (0)$, i.e., n = 1 and the result follows.
- Suppose m ≥ 2 and assume that the theorem is true for ay module having some composition series of length atmost m-1.

Consider two cases.

Case 1: $M_1 = N_1$

Now M_1 and N_1 have their composition series,

$$M_1 \supset M_2 \supset \cdots M_m = (0)$$

$$N_1 \supset N_2 \supset \cdots N_n = (0)$$

Since the first one is of length m-1, we get by induction that

a. $m - 1 = n - 1, i.e., m = n$ and
b. $\forall\, 1 \leq i \leq m - 1, \exists\, j = j(1), 1 \leq j \leq n - 1$ such that $M_i / M_{i+1} \cong N_j / N_{j+1}$ and vice versa.

Since $M / M_1 \cong N / N_1$, the result follows.

Case 2: $M_1 \neq N_1$

Let $M' = M_1 + N_1$ which is a submodule of M containing the submodules M_1 and N_1.

But M_1 and N_1 both are maximal.

Hence $M' = M$.

$$\left.\begin{aligned} M/M_1 &= (M_1 + N_1)/M_1 \cong N_1/(M_1 \cap N_1) = N_1/K = simple \\ M/N_1 &= (M_1 + N_1)/N_1 \cong M_1/(M_1 \cap N_1) = M_1/K = simple \end{aligned}\right\} \quad (1)$$

Where K = $M_1 \cap N_1$, which is a submodule of a module of finite length M and thus K is also of finite length.

Hence K has composition series, say $K = K_0 \supset K_1 \supset \cdots \supset K_r = (0)$

Thus, we get four composition series for module M = $M_1 + N_1$.

i.e., $M = (M_1 + N_1) \supset M_1 \supset K \supset K_1 \supset \cdots K_r = (0)$,

$M = (M_1 + N_{1)} \supset M_1 \supset \cdots M_m = (0)$

$M = (M_1 + N_1) \supset N_1 \supset \cdots N_n = (0)$

$M = (M_1 + N_1) \supset N_1 \supset K \supset K_1 \supset \cdots K_r = (0)$

Of these, the first two are equivalent by case (1) and similarly the last two are equivalent and the first and third are equivalent by (1) above and hence we get second and third are equivalent.

Hence proved.

CONCLUSION

In mathematics, a module is one of the fundamental algebraic structures used in abstract algebraic. A module over a ring is a generalization of the notion of vector space over a field where in the corresponding scalars are the element of an arbitrary given ring with identity and a multiplication is defined between element of the ring and element of the module.

Modules can be quite a bit more complicated than vector spaces, for instances, not all modules have a basis, and even those that do, free modules, need not have a unique rank if the underlying ring does not satisfy the invariable basis number condition, unlike vector spaces, which always have a basis whose cardinality is then unique. Thus, modules are very closely related to the representation theory of groups. They are also one of the central notions of commutative algebra and homological algebra, and are used widely in algebraic geometry and algebraic topology.

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