LIE ALGEBRA

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NAMITHA PADMAJAN

Reg No. 200011014782

Under The Supervision of

Mrs. KARTHIKA V

DEPARTMENT OF MATHEMATICS

Bharata Mata College, Thrikkakara

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Place: Thrikkakara

Date: Namitha Padmajan

CERTIFICATE

This is to certify that the dissertation entitled "LIE ALGEBRA" submitted by Namitha Padmajan is a record of work done by the candidate during the period of her study under my supervision and guidance.

Mrs. Karthika V
Department of Mathematics
Bharata Mata College,
Thrikkakara.

Place : Thrikkakara
Date :

DECLARATION

I hereby declare that the project report entitled **"LIE ALGEBRA"** submitted for the M.Sc. Degree is my original work done under the supervision of Mrs. Karthika V and the project has not formed the basis for the award of any academic qualification fellowship or other similar title of any other university or board.

Place:

Date: Namitha Padmajan

CONTENTS

LIE ALGEBRAS

INTRODUCTION

Lie algebra (pronounced as "LEE", named in the honour of Norwegian mathematician marius sophus lie(1842 - 1899) is an algebraic structure inmathematics whose main use lies in studying geometric objects such as Lie groups and differentiable manifolds. Lie algebras were introduced to study the concept of infinitesimal transformations. The term "Lie algebra"(after Sophus Lee) was introduced by Hermann Weyl in the 1930s. In the older texts, the name "infinitesimal group" is used.

Advances in the theory of Lie algebra developed by Sophus Lie,have long enriched mathematics , particularly in the area of group theory. In 1952 Lie algebra was used to solve one of the most famous problems in mathematics, the so called Hilbert's fifth problem , posed in the year 1900by David Hilbert (1862- 1943) .In addition to its use solving this problem, Lie algebra has been used to gain a better understanding of properties of many dimensional surfaces in general, helping to advance the mathematical discipline of topology as well.

ABSTRACT

Here we conduct a study on Lie Algebra. Its basic concepts, examples, Types of Lie Algebras. First chapter is an Introduction to Lie Algebra here we see the definitions and few examples and come across Lie Algebra of derivations and Abstract Lie Algebra. Second chapter deals with Ideals and Homomorphism, construction with the Ideals and Homomorphism representation, and then we come to next chapter, Chapter 3 which is about Solvability and Nilpotency where we encounter an important theorem "Engels Theorem". In 4^{th} chapter we discuss an important topic Semi Simple Lie Algebras which plays a vital role in the study of Lie Algebra, here first we learn "Lie Theorem", Jordan-Chevalley decomposition, killing form, Criterion for semi simplicity, Cartan's Criterion, Abstract Jordan decomposition and finally the complete reducibility of representations. This chapter paves ways to further study of Lie Algebra.

Chapter 1

INTRODUCTION TO LIE ALGEBRA

1.1. Basic Concepts

Definition 1.1.1

Definition. A vector space L over a field F, with an operation Lx L→L, denoted (x, y) ↦ [xy] and called the bracket or commutator of x and y, is called a Lie algebra over F if the following axioms are satisfied:

(L1) The bracket operation is bilinear.

(L2) [xx] = 0 for all x in L.

(L3) [x [yz]] + y[zx]] + [z[xy]] = 0 (x, y, z ∈ L)

Axiom (L3) is called the Jacobi identity

A Lie algebra is called real or complex when the vector space is respectively real or complex.

Remark 1.1.1

For any x, Є L ,

$[x+y, x+y] = [x, x] + [x, y] +[y, x] +[y, y]$ .... (1)

But by L2 we have,

$[x, x] = 0$, $[y, y] = 0$, $[x+y, x+y] = 0$

Therefore (1) implies, $[x, y] = - [y, x]$ for all x, Є L ....(L2')

Thus, Lie bracket is anticommutative.

Definition 1.1.2

Two Lie algebras L, L' over F are isomorphic if there exists a vector space isomorphism ϕ: L→L' satisfying

$\phi ([x, y]) = [\phi (x) , \phi (y)]$

for all x, y in L (and then ϕ is called an isomorphism of Lie Algebras)

Definition 1.1.4

A subspace K of L is called a subalgebra if [xy] Є K whenever x, y Є K:)

Example 1.1.1

Let V is a finite dimensional vector space over F. Let End V denote set of all linear transformations from V to V. End V is a vector space over F with dimension $n2$ and it is a ring relative to the usual product operation. Then End V with the operation $[x, y] = xy - yx$ called the bracket of x and y is a Lie algebra over F. This Lie algebra is called the **general linear algebra** denoted by $\mathfrak{gl}$ (V) and it can also be identified with the set of all $n \times n$ matrices over F, denoted by $\mathfrak{gl}$ $(n$,F)

Any sub algebra of a Lie Algebra $\mathfrak{gl}$(V) is called a Linear Lie Algebra

Example 1.1.2

Let dim V=ℓ+1. Denote by $\mathfrak{sl}$(V) or by $\mathfrak{sl}(\ell+1,F)$ the set of endomorphisms of V having trace zero .(Recall that trace of a matrix is the sum of its diagonal entries, this is independent of the choice of basis for V, hence makes sense for an endomorphism of V). Since Tr(xy) = Tr(yx), and Tr(x+y) = Tr(x) +Tr(y) $\mathfrak{sl}$(V) is a subalgebra of $\mathfrak{gl}$(V) called the **Special linear algebra**.

Example 1.1.3

Let dim V = 2ℓ, with basis ($v_{1,2,...,}v_{2\ell}$).Define a nondegenerate skew – symmetric form f on V by the matrix $S = \begin{pmatrix} 0 & I\ell \\ -I\ell & 0 \end{pmatrix}$ The set of all endomorphisms x of V satisfying $f(\ x(v)\ ,\ w\ ) = -\ f(\ v,\ x(w)\ )$ is called the **Symplectic algebra** denoted by $\mathfrak{sp}$(V) or $\mathfrak{sp}(2\ell$,F). dim $\mathfrak{sp}(2\ell$,F) = $2\ell^2 + \ell$.

Example 1.1.4

An $n \times n$ matrix A is **orthogonal** denoted by O (n) if the column vectors that make up A are orthonormal, that is, if

$$\sum_{i=1}^{n} A_{ij} A_{ik} = \delta_{jk}$$

Equivalently, A is orthogonal if it preserves inner product, namely if

$< x, y > = < Ax, Ay >$ for all $x, \in R^n$.

Let dim V = 2ℓ +1 be odd and take f to be the nondegenerate symmetric bilinear form on V

whose matrix is $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I\ell \\ 0 & I\ell & 0 \end{pmatrix}$ The set of all endomorphisms of V satisfying

$f(x(v) ,w) = - f (v , x (w))$ is called the **Orthogonal algebra** $\mathfrak{o}$(V) or $\mathfrak{o}(2\ell + 1 ,F)$.
dim $\mathfrak{o}$(V) = $2\ell^2 +\ell$.

Similar to SL(n,C), the special orthogonal group, denoted by SO(n), is defined as subgroup of O(n) whose matrices have determinant 1. Again, this is a matrix Lie group.

Example 1.1.5

Let (n,F) be the upper triangular matrices in $\mathfrak{gl}$(n,F) (A matrix A is said to be upper triangular if x_{ij} = o whenever i > j .). This is Lie algebra with same Lie bracket as (n,F).

1.2 Lie Algebras of Derivations

[By an F-algebra (not necessarily associative) We simply mean a vector space V over F endowed with bilinear operation VxV→V,

By a derivation ofVmean a linear map δ : V→V satisfying the familiar product rule δ(ab= a δ(b)+ δ(a)b. It is easily checked that the collection Der V of all derivations of V is a vector sub space of End V.

The commutator [δ, δ'] of two derivations is again a derivation. So Der(V) is a subalgebra of $\mathfrak{gl}$(V).

Example 1.2.1.

Let x, y ∈ End(V), and δ, δ' ∈ Der(V).By definition of the commutator,

[δ, δ'] = (δδ' − δ'δ):

• Der(V) is a vector subspace of End(V):

δ[x, y] = δ(xy − yx) = δ(xy) − δ(yx)

= xδ(y) + δ(x)y − (yδ(x) + δ(y)x)

= δ(y)x − yδ(x) + xδ(y) − δ(y)x

= [δ(x), y] + [x, δ(y)]

The commutator $[\delta, \delta']$ of two derivations $\delta, \delta' \in$ Der(V) is again a derivation.

$([\delta, \delta'](x))y + x([\delta, \delta'](y)) = ((\delta\delta' - \delta'\delta)(x))y + x(\delta\delta' - \delta'\delta)(y)$

$= (\delta\delta'(x) - \delta'\delta(x))y + x(\delta\delta'(y) - \delta'\delta(y))$

$= \delta\delta'(x)y - \delta'\delta(x)y + x\delta\delta'(y) - x\delta'\delta(y)$

$= \delta(\delta'(x)y + x\delta'(y)) - \delta'(\delta(x)y + x\delta(y))$

$- \delta'(\delta(x)y) - \delta(x)\delta'(y) - \delta'(x)\delta(y) - x\delta'\delta(y)$

$= \delta(\delta'(x)y + x\delta'(y)) - \delta'(\delta(x)y + x\delta(y))$

$= \delta(\delta'(xy) - \delta'(\delta(xy))$

$= [\delta,\delta'](xy)$

Definition 1.2.1

Given x L, ∈ the map y → [x, y], is an **endomorphism** of L, denoted adx, where adx is an inner derivation. Derivations of the form [x[yz]] = [[xy]z] +[y[xz]] are inner. All others are outer.

Example 1.2.2.

By example 2.3.1, the collection of derivations, Der(V), satisfies skew symmetry. By definition 2.3.2, Der(V) satisfies the Jacobi identity. Therefore Der(V) defines a Lie algebra.

Definition 1.2.2

The map L → DerL sending x to adx is called the **adjoint representation** of L.

Definition 1.2.3

If L is any Lie algebra with basis x1, ..., xn, then the multiplication table for L is determined by the **structural constants**, the set of scalars $\{b_{ijk}\}$ such that

$[x_i, x_j] = \Sigma b_{ijk} x_k$ for all k =1 to n.

1.3 Abstract Lie Algebras

Sometimes it is desirable however, to contemplate Lie Algebras abstractly. For Example if L is an arbitrary finite dimensional cector space over F, We can view L as a Lie Algebra by setting [xy] = 0 for all $x,y \in L$. Such an algebra having trivial Lie multiplication, is called **Abelian.**(because in the linear case[x,y]=0 just means that x and y commute). If L is any Lie algebra, with basis $x_{1,}x_{2........}x_{n.}$ It is clear that the entire multiplication table of L can be recovered from the **structure constants** a_{ij}^K which occur in the expressions

$[x_{i,} x_j] = \Sigma a_{ij}^k x_k$

Example 1.3.1.

Given L as a Lie algebra, with [xy] = 0 for all $x, y \in L$,

$[xy] = (xy - yx) = 0 \Leftrightarrow xy = yx$

Therefore L under trivial Lie multiplication, is abelian. Similarly, [yx] = 0.

Chapter 2
IDEALS & HOMOMORPHISMS

2.1 Ideals

A subspace I of a Lie algebra L is called an ideal of L if $x \in L$, $y \in I$ together imply $[x, y] \in I$. (since $[xy] = -[yx]$, the condition could just as well be written : $[y,x] \in I$)

Example 2.1.1

(1) Obviously 0 and L itself are ideals in L.

(2) (**centre**) A less trivial example is the center
$Z(L) := \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}$

(3) (**derived algebra**) Another important example is the derived algebra of L denoted by
$[L, L] := \mathrm{Span}(\{[x, y] \mid x, y \in L\})$
which is analogous to the commutator subgroup of a group. It consists of all linear combinations of commutators [x,y], and is clearly an ideal.

(4) (**sum**) If I, J are two ideals of a Lie algebra L,
then $I+J =\{ x+y \mid x\in I, y\in I\}$ is also an ideal.

Correspondence between Ideals

Suppose that I is an ideal of the Lie algebra L. There is a bijective correspondence between the ideals of the factor algebra L/I and the ideals of L that contain I. This correspondence is as follows. If J is an ideal of L containing I, then J/I is an ideal of L/I. Conversely, if K is an ideal of L/I, then set $J := \{z \in L : z +I \in K\}$. One can readily check that J is an ideal of L and that J contains K. These two maps are inverses of one another.

Example 2.1.2

Suppose that L is a Lie algebra and I is an ideal in L such that L/I is abelian. In this case, the ideals of L/I are just the subspaces of L/I. By the ideal correspondence, the ideals of L which contain I are exactly the subspaces of L which contain I.

Definition 2.1.1

If L = [L, L], then we call L a perfect Lie algebra.

Definition 2.1.2

A Lie algebra L is called simple if L has no ideals except itself and 0, and if moreover [L, L] $\neq$ 0

Definition 2.1.3

If L and L' are Lie algebras over F, then we can define a Lie algebra structure on the direct sum L L' as [x + x', y + y'] = [x, y] + [x', $\oplus$ y'] for x, y $\in$ L, x', y'$\in$ L'.

The Lie algebra L $\oplus$ L' is called the direct product of L and L'.

Constructions with Ideals

Suppose that I and J are ideals of a Lie algebra L. There are several ways we can construct new ideals from I and J. First we shall show that I ∩ J is an ideal of L. We know that I ∩ J is a subspace of L, so all we need check is that if x $\in$ L and y $\in$ I ∩ J, then [x, y] $\in$ I ∩ J: This follows at once as I and J are ideals.

Exercise 2.1.1

Show that I+ J is an ideal of L where

I + J:= {x + y : x $\in$ I, y $\in$ J}.

We can also define a product of ideals. Let

[I,J] := Span{[x, y] : x $\in$ I,y $\in$ J}.

Proof

We claim that [I,J] is an ideal of L. Firstly, it is by definition a subspace.Secondly, if x $\in$ I, y $\in$ J, and u $\in$ L, then the Jacobi identity gives

[u, [x, y]] = [x, [u, y]] + [[u, x], y].

Here [u, y] $\in$ J as J is an ideal, so [x, [u, y]] $\in$ [I,J]. Similarly, [[u, x], y] [I,J].

Therefore their sum belongs to [I,J].

A general element t of [I,J] is a linear combination of brackets [x, y] with x $\in$ I, y $\in$ J, say t = Σci , yi], where the ci are scalars and xi $\in$ I and yi $\in$ J. Then, for any u $\in$ L, we have

[u, t] = u,Σ ci[xi, yi] = Σci[u, [xi, yi]],

where [u, [xi, yi]] ∈ [I,J] as shown above.Hence [u, t] ∈ [I,J] and so [I,J] is an ideal of L.

Definition 2.1.4

- The normalizer of a subalgebra (or just subspace) K of L is defined by NL(K) = {x ∈ L | [x,K] ⊆ K}.
- If K = NL(K), we call K self-normalizing
- The centralizer of a subset X of L is
- CL (X) := {x ∈ L | [x, X] =0}.

Definition 2.1.5

Let L be a Lie algebra over a field F and K an ideal of L. Then the **quotient space** L/K: = {x + K | x ∈ L} where x + K: = {x + k | k ∈ K} for x ∈ L . The bracket operation on L/K is then defined by [x + K, y + K]:= [x, y] + K. This is well-defined because K is an ideal, and it inherits all the axioms directly from L.

2.2. Homomorphisms and Representations

A linear transformation ф: L → L ' is called a **homomorphism** if ф($[x, y]$) = [ф(x) , ф (y)] for all x,y Є L . ф is called a **monomorphism** if Kernal(ф) = 0 , an **epimorphism** if Image (ф) = L ' , an **isomorphism** if it is both monomorphism and epimorphism.

A **representation** of a Lie algebra L is a homomorphism ф: L → gl (V) where V is a vector space over F.

The adjoint representation ad:L → gl(L) is an example of representation of a Lie algebra Clearly , ad is a linear transformation.

Consider, [ad$x$, ad] (z) = ad x ad y (z) – ad y ad x (z)

= ad x ($[y, z]$) – ad y ($[x, z]$)

= $[x, [y, z]]$ – $[y, [x, z]]$

= $[x, [y, z]]$ + $[[x, z], y]$ (by L2)

= $[[x, y], z]$ (by L3)

= ad $[x, y]$ (z)

Thus, ad preserves the bracket.

It consists of all x Є L for which ad x = 0. i.e., for which [, y]= 0 (for all y ЄL).

So Ker (ad) = Z (L)

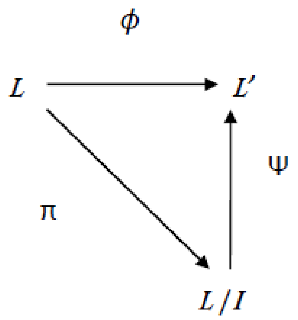
Proposition 2.2.1

a) If ɸ :L → L ' is a homomorphism of Lie algebras , then L / ker ɸ ≅ Im ɸ.

If I is any ideal of L induced in ker ɸ, there exist a unique homomorphism

Ψ : L /I →L' making the following diagram

commute. (π = canonical map):



b) If I and J are ideals of L such that I ⊂ J , then I /J is an ideal of L /I and (L /I)/(J /I) is mutually isomorphic to L /J.

c) If I , J are ideals of L , there is a natural isomorphism between (I +J)/J and I /(I ∩ J).

Proof

a) Let ϕ : L → L' be a homomorphism of Lie algebras. Our first goal is to show given I is any ideal of L included in Kerϕ, there exists a unique homomorphism ψ : L/I → L'. Let Ik,Il ∈ L/I.To show ψ is a group isomorphism, note that

j = j' l. If j ∈ Il then ϕ(j' l) = ϕ(j')ϕ(l), since ϕ is a homomorphism. Therefore ψ is well defined. For all j, k ∈ I:

ψ(IkIl) = ψ(Ikl) = ϕ(kl) = ϕ(k)ϕ(l) = ψ(Il)ψ(Ik)

Therefore ψ is a homomorphism. Let kerϕ = I,

ϕ(Il)=1 ⇐⇒ ϕ(l)=1 ⇐⇒ l∈kerϕ ⇐⇒ l∈I ⇐⇒ Il = I

b) Fist note: if i ∈ I ∩ J, then by definition of the intersection, [i, x] ∈ I and

$[i, x] \in J$, so $[i, x] \in I \cap J$. Therefore, $I \cap J$ satisfies the definition of an ideal. Given $x + I \in L/I$ and $j + I \in J/I$, we have:

$[x + I, j + I] = [x, j] + [x, I] + [I, j] + [I, I] = [x, j] + I \in J/I$

Where $[x, j] \in L$. Therefore, J/I is an ideal of L/I. Additionally, given x + I,
$y + I \in L/I$, $x + I = y + I \mod J/I$ implies that $(x - y) + I \in J/I$ and therefore $x - y = j \in J$.
Thus x and y are equivalent mod J in L.

c) Let $i_1 + j_1, i_2 + j_2 \in I+J$. If $i_1 + j_1 = i_2 + j_2 \mod J$, then $(i_1 - i_2) = j_2 - j_1 = j \in J$, but
$i_1 - i_2 \in I$. Therefore $i1 - i_2 \in I \cap J$

Definition 2.2.1

A **representation** of a Lie algebra L is a homomorphism $\phi : L \to gl(V)$ where V is a vector space over F where V is a finite dimensional vector space over F. Sometimes we say (V, ϕ) is a representation of L.

Proposition 2.2.2

Any simple Lie algebra is isomorphic to a linear Lie algebra.

Proof.

If L is simple, then $Z(L) = 0$, so that $ad : L \to gl(L)$ is a monomorphism. Hence L is isomorphic to a subalgebra of gl(L).

Definition 2.2.2

An **automorphism** of L is an isomorphism of L onto itself. Aut(L) denotes the group of all automorphisms of L.

Chapter 3

SOLVABLE AND NILPOTENT LIE ALGEBRAS

3.1 Solvability

Lemma 3.1.1

Suppose that I is an ideal of L. Then L/I is abelian if and only if I contains the derived algebra L'.

Proof

The algebra L/I is abelian if and only if for all x, y $\in$ L we have

[x +I,y + I]=[x, y] + I= I

or, equivalently, for all x, y $\in$ L we have [x, y] $\in$ I. Since I is a subspace of L,this holds if and only if the space spanned by the brackets [x, y] is contained in I; that is, L'$\subseteq$ I.

This lemma tells us that the derived algebra L is the smallest ideal of L with an abelian quotient.

Definition 3.1.2

The **derived series** of a Lie algebra L is a sequence of ideals of L where

$L^{(0)} = L$, $L^{(1)}= [LL]$, $L^{(2)} = [L^{(1)}L^{(1)}]$,........ $L^{(i)} = [\ L^{(i-1)})L^{(i-1)}\ ]$

This gives a descending sequence of ideals,

$L^{(0)} \supseteq L^{(1)} \supseteq L^{(2)}) \supseteq \cdot \ \cdot \ \cdot$

Definition 3.1.3

The Lie algebra L is said to be **solvable** if for some m ≥ 1 we have $L^{(m)} = 0$

A simple algebra is not solvable (since $L^{(i)} = L \ \forall_{\ i}$)

Lemma 3.1.4

L is a Lie algebra with ideals

$L = I_0 \supseteq I_1 \supseteq ... \supseteq I_{m-1} \supseteq I_m = 0$

Such that I_{k-1}/I_k is abelian for $1 \leq k \leq m$, then L is solvable.

Proof

We shall show that $L^{(k)}$ is contained in I_k for k between 1 and m. Putting k = m will then give $L^{(m)} = 0$.

Since L/I_1 is abelian, we have from Lemma 4.1.1 that L' ⊆ I_1.For the inductive step, we suppose that $L^{(k-1)} \subseteq I_{k-1}$, where $k \geq 2$. The Lie algebra I_{k-1}/I_k is abelian. Therefore by Lemma 3.1.4, this time applied to the Lie algebra I_{k-1}, we have $[I_{k-1}, I_{k-1}] \subseteq$ Ik. But $L^{(k-1)}$ is contained in I_{k-1} by our inductive hypothesis, so we deduce that $L^{(k)} = [L^{(k-1)}, L^{(k-1)}] \subseteq [I_{k-1}, I_{k-1}]$, and hence $L^{(k)} \subseteq I_k$

Proposition 3.1.5

Let L be a Lie algebra.

(a) If L is solvable, then so are all subalgebras and homomorphic images of L.

(b) If I is a solvable ideal of L such that L/I is solvable, Then L itself is solvable.

(c) If I and J are solvable ideals of L, then so is I + J

Proof

(a) From the definition, if K is a subalgebra of L, then $K^{(i)} \subset L^{(i)}$. Similarly, if φ : L → M is an epimorphism, an easy induction on i shows that $\varphi(L^{(i)}) = M^{(i)}$.

(b) Say $(L/I)^{(n)} = 0$. Applying part (a) to the canonical homomorphism $\pi : L \to L/I$, we get $\pi(L^{(n)}) = 0$, or $L^{(n)} \subset I = \text{Ker } \pi$. Now if $I^{(m)} = 0$, the obvious fact that $(L^{(i)})^{(j)} = L^{(i+j)}$implies that $L^{(n+m)} = 0$ (apply proof of part (a) to the situation $L^{(n)} \subset I$).

(c) One of the standard homomorphism theorems (proposition 3.2.3 c) yields an isomorphism between (I+J)/J and I/ (I ∩J). As a homomorphic image of I, the right side is solvable, so (I+ J)/J is solvable. Then so is I + J, by part

(b) applied to the pair I+ J, J.

Definition 3.1.6

As a first application, Let L be an arbitrary Lie Algebra and let S be a maximal solvable ideals (that is, one included in no larger solvable ideals) If I is any other solvable ideals of L, then part (c) of the proposition forces S+I = S(by maximality), or I⊂ S. This proves the existence of a unique maximal solvable ideal, called the **radical** of L and denoted by **Rad L.** In case Rad L = 0, L is called **semisimple.** For example, a simple algebra is semisimple: L has no ideals except itself and 0, and L is nonsolvable. Also L = 0 is semisimple.

semisimple if it has no non-zero solvable ideals or equivalently if radL = 0.

3.2 Nilpotency

Definition 3.2.1

The **descending central series** is a sequence of ideals of L defined as

$L^{\mathbf{0}} = L$, $L^{\mathbf{1}} = [LL]$, $L^{\mathbf{2}} = [\ LL^{\mathbf{1}}\ ]$,......... $L^{\mathbf{i}} = [L^{\mathbf{i-1}}\ L^{\mathbf{i-1}}]$

Definition 3.2.2

The Lie algebra L is said to be **nilpotent** if for some m ≥ 1 we have Lm = 0.

Examples

For example, any abelian algebra is nilpotent

Clearly, $L^{(i)} \subseteq L^{i}$ for all i, so nilpotent algebras are solvable. The converse is false.

Proposition 3.2.3

Let L be a Lie algebra

(a). If L is nilpotent, then so are all subalgebras and homomorphic images of L.

(b). If L/Z(L) is nilpotent, then so is L.

(c). If L is nilpotent and nonzero, then Z(L) ≠ 0.

Proof

(a) Let K be a subalgebra, then by definition Ki ⊂ Li. Similarly, if $\phi : L \to M$ is an epimorphism, we can show, by induction on i, that ϕ(Li) = Mi

(b) Let Ln ⊂ Z(L), then $L^{n+1} = [L, L^{n}]$ {By definition of the descending central series}

= [L, Z(L)] = 0 {By definition of the center}

(c) The last nonzero term of the descending central series is central, i.e if

Ln = 0 and L^{n-1} 6= 0, then $[L^{n-1}, L] = 0$ and this implies that Z(L) $\supseteq L^{n-1}$ 6= 0.

Example 3.2.4

Let I be an ideal of L. Then each member of the descending central series of I is also an ideal of L.

Proof

Since I is an ideal of L, I 0 = I is also an ideal of L. Assume $I^n=[I, I^{n-1}]$ is an ideal of L. Let $x \in L, y \in I, z \in I^n$

$[x[yz]] = -[y[zx]] - [z[xy]]$. {By the Jacobi identity}

$\in [y, I^n] + [z, I]$ {Since $[z, x] \in I^n$, $[x, y] \in I$}

$\in [I,I^n] + [I^n,I]$ {Since $y \in I, z \in I^n$}

$= I^{n+1} + I^{n+1}$

$= I^{n+1}$

Therefore I^{n+1} is an ideal, then each member of the descending central series of I is an ideal of L.

3.3 Proof of Engel's Theorem

Lemma 3.3.1

Let A be a nilpotent operator on a vector space V, then

1. There exists a non zero $v \in V$ such that $Av = 0$
2. ad_A is a nilpotent operator on gl(v).

Theorem 3.3.2

Let L be a subalgebra of gl(v), with V finite dimensional. If L consists of nilpotent endomorphisms and $V \neq 0$, then there exists a nonzero $v \in V$ for which $L.v = 0$.

Proof. Use induction on dim L, the case dim L = 0 (or dim L = 1) being obvious. Suppose $K \neq L$ is any subalgebra of L. According to Lemma 3.2, K acts (via ad) as a Lie algebra of nilpotent linear transformations on the vector space L, hence also on the vector space L/K. Because dim K < dim L, the induction hypothesis guarantees existence of a vector $x+K \neq K$ in L/K killed by the image of K in gl(L/K). This just means that $[yx] \in K$ for all $y \in K$, whereas $x \notin K$. In other words, K is properly included in $N_L(K)$ (the normalizer of K in L, see (2.1)). Now take K to be a maximal proper subalgebra of L. The preceding argument forces $N_L(K) = L$, i.e., K is an ideal of L. If dim L/K were greater than one, then the inverse image in L of a one dimensional subalgebra of L/K (which always exists) would be a proper subalgebra properly containing K, which is absurd; therefore, K has codimension one. This allows us to write $L = K+ Fz$ for any $z \in L-K$.

By induction, $W = \{v \in V|K.v = 0\}$ is nonzero. Since K is an ideal, W is stable under L: $x \in L$, $y \in K$, $w \in W$ imply $yx.w = xy.w —[x, y].w = 0$. Choose $z \in L — K$ as above, so the nilpotent endomorphism z (acting now on the subspace W) has an eigenvector, i.e., there exists nonzero v e W for Which $z.v = 0$. Finally, $L.v = 0$, as desired.

Proof of Engel's Theorem. We are given a Lie algebra L all of whose elements are ad-nilpotent; therefore, the algebra ad $L \subset gI(L)$ satisfies the hypothesis of Theorem 3.3. (We can assume $L \neq 0$.) Conclusion: There exists $x \neq 0$ in L for which $[Lx] = 0$, i.e., $Z(L) \neq 0$. Now L/Z(L) evidently consists of ad-nilpotent elements and has smaller dimension than L. Using induction on dim L, we find that L/Z(L) is nilpotent. Part (b) of Proposition 3.2 then implies that L itself is nilpotent.

Theorem 3.3.3 Engel I

Let V be a vector space; let L be a sub Lie algebra of the general linear Lie algebra gl(V), consisting entirely of nilpotent operators. Then L is a nilpotent Lie algebra.

Proof

By Theorem 4.3.2, if we apply the dual representation (inverse transpose) of L on $V^{t;}$ the operators are nilpotent. Let $\lambda \neq 0$ be a function on V that is annulled by L, then the space $(L \cdot v)$ spanned by all Xv with $X \in L$, is a proper subspace of V , and in fact, is in the kernal of λ where $\lambda(Xv) = X^t\lambda(v) = 0$.

Recall $(L \cdot v)$ is invariant under L, so we can iterate the argument.

Let m = dimV , then the abbreviated iteration $X_1 \cdot X^2...X_m$, vanishes, since for every X_i, the dimension of V is decreased by at least 1 where any long bracket $[X_1X_2...X_k]$ expands by the bracket operation into a sum of products of kX's.

Theorem 3.3.4 Engel II

If all elements of L are ad-nilpotent, then L is nilpotent.

Proof

Given a Lie algebra of L having only ad-nilpotent elements, since the adoint is a linear transformation, $ad_L \subset gl(L)$, which satisfies the hypothesis of Theorem 3.3.2 when $L \neq 0$.

Thus there exists an $x \in L$ such that $[L, x] = 0$, which implies that the center of L is nonzero. Therefore L/Z(L) consists of adnilpotent elements, where dim adL/Z(L) < dimL.
Using an induction argument on the dimension of L, similar to Theorem 3.3.2, it follows that L/Z(L) is nilpotent. Therefore, by Proposition 4.2.3 (part2.), if L/Z(L) is nilpotent, then so is L.

Corollary 3.3.5

Let L be nilpotent, and K be an ideal of L. Then ifk 6= 0, k ∩ Z(L) 6= 0

Proof

Since K is an ideal of L, L induces a linear transfomation on K via the adjoin representation, therefore there exists $x \in$ K(k 6= 0). Therefore $[L, x] = 0$ by definition of nilpotency, and thus $x \in K \cap Z(L)$ as desired.

Chapter 4

SEMI SIMPLE LIE ALGEBRAS

Theorem. Let L be a solvable subalgebra of gl(t), V finite dimensional. If V 0 0, then V contains a common eigenvector for all the endomorphisms in L.

Proof Use induction on dim L, the case dim L = 0 being trivial. We attempt to imitate the proof of Theorem 3.3 (which the reader should review at this point). The idea is (1) to locate an ideal K of codimension one,

(2) to show by induction that common eigenvectors exist for K, (3) to verify that L stabilizes a space consisting of such eigenvectors, and (4) to find in that space an eigenvector for a single z E L satisfying L = K+ Fz.

Step (1) is easy. Since L is solvable, of positive dimension, L properlyincludes [LL]. L/ [LL] being abelian, any subspace is automatically an ideal.

Take a subspace of codimension one, then its inverse image K is an ideal of codimension one in L (including [LL]).

For step (2), use induction to find a common eigenvector v e V for K (K is of course solvable; if K= 0, then L is abelian of dimension 1 and an eigen- vector for a basis vector of L finishes the proof.) This means that for x ∈ K

x.v = A(x)v, A: *K*F some linear function. Fix this A, and denote by W the subspace {w E Vix.w = A(x)w, for all x E *K};* so W # 0.

Step (3) consists in showing that *L* leaves W invariant. Assuming for the moment that this is done, proceed to step (4): Write *L = K+Fz,* and use the fact that F is algebraically closed to find an eigenvector v_o e W of *z* (for some eigenvalue of *z).* Then v_c, is obviously a common eigenvector for *L* (and A can be extended to a linear function on *L* such that $x.v_o = A(x)v_o$, x E *L).*

It remains to show that *L* stabilizes W. Let w E W, *x* e *L.* To test whether or not x.w lies in W, we must take arbitrary y E *K* and examine *yx.w = xy.w—[x, y].w = A(y)x.w* — A([x, *y])w.* Thus *we have to prove that A([x, y])* = 0. For this, fix w E W, x E *L.* Let n > 0 be the smallest integer for which w, x.w, ... , x".w are linearly dependent. Let W, be the subspace of *V* spanned by w, x.w, , ^{1}.w (set W_o = 0), so dim W_n = *n*,

Wn = Wn+1 (i 0) and *x* maps W,, into W. It is easy to check that each y E *K* leaves each *W,* invariant. Relative to the basis w, x.w, , x^{11-1}.w of W,,, we claim that y E *K* is represented by an upper triangular matrix whose diagonal entries equal A(y). This follows immediately from the congruence:

(*) *yx'.w A(y)*^{i}*.w* (mod W_i),

which we prove by induction on *i,* the case *i* = 0 being obvious. Write $yx^{i}.w$ = yxx^{i-1} .w=[x, y]x^{l-} .w. By induction, $yx^{j-1}.w$ = A(y)xl" i.w+ w' (w' e W,_1); since x maps W,_, into W, (by construction), (*) therefore holds for all *i.*

According to our description of the way in *which y e K acts on W,,, Tr_{w}.(y)* = nA(y). In particular, this is true for elements of *K* of the special form [x, y] (x as above, y in *K).* But x, y both stabilize W,,, so [x, y] acts on W,, as the *commutator* of two endomorphisms of W,,; its trace is therefore 0. We conclude that *nA([x, y])* = 0. Since char ***F*** = 0, this forces *A([x, y])* = 0, as required. 0

Corollary A (Lie's Theorem). Let L be a solvable subalgebra of g l(V), dim V = n < co. Then L stabilizes some flag in V (in other words, the matrices of L relative to a suitable basis of V are upper triangular).

Proof Use the theorem, along with induction on dim V. 0

More generally, let *L* be any solvable Lie algebra, 0: *L* gl(V) a finite dimensional representation of *L.* Then *fl)(L) is* solvable, by Proposition 3.1(a), hence stabilizes a flag (Corollary **A).** For example, if is the adjoint representation, a flag of subspaces stable under *L is* just a chain of ideals of *L*, each of codimension one in the next. This proves:

Corollary B. Let L be solvable. Then there exists a chain of ideals of L, 0 = L_0 c L_1 ⊂ ... ⊂ L_n = L, such that dim L_i =i

Corollary C. Let L be solvable. Then x E [LL] implies that ad_L x is nilpotent. In particular, [LL] is nilpotent.

Proof Find a flag of ideals as in Corollary B. Relative to a basis (x….xn) of L for which (x_1,…. x_i) spans Li, the matrices of ad L lie in t(n, F).

Therefore the matrices of [ad L, ad L] = ad, [LL] lie in n(n, F), the derived algebra of t(n, F). It follows that ad, x is nilpotent for x E [LL]; a fortiori x is nilpotent, so [LL] is nilpotent by Engel's Theorem.

4.2. Jordan-Chevalley decomposition

In this subsection only, char F may be arbitrary. We digress in order to introduce a very useful tool for the study of linear transformations. The reader may recall that the Jordan canonical form for a single endomorphism x over an algebraically closed field amounts to an expression of x in matrix' form as a sum of blocks

$$\begin{bmatrix} a & 1 & & & 0 \\ & a & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & a \end{bmatrix}$$

Since diag (a,…a) commutes with the nilpotent matrix having one's just above the diagonal and zeros elsewhere, x is the sum of a diagonal and a nilpotent matrix which commute. We can make this decomposition more precise, as follows.

Call x ∈ End V (V finite dimensional) **semisimple** if the roots of its minimal polynomial over F are all distinct. Equivalently (F being algebraically closed), x semisimple if and only if x is diagonalizable. We remark that two commuting semisimple endomorphisms can be simultaneously diagonalized; therefore, their sum or difference is again semisimple (Exercise 5). Also, if x is semisimple and maps a subspace W of V into itself, then obviously the restriction of x to W is semisimple.

Proposition. Let V be a finite dimensional vector space over F, x ∈ End V.

(a) There exist unique x_5, x_n ∈ End V satisfying the conditions: $x = x_s+x_n$ x, is semisimple, x_s is nilpotent, x_s, and x_n commute.

(b) There exist polynomials p (T), q (T) in one indeterminate, without constant term, such that x, = p(x),x. = q(x). In particular, x, and x,, commute with any endomorphism commuting with x.

(c) If A c B c V are subspaces, and x maps B into A, then x, and x,, also map B into A.

The decomposition x = xs+x,, is called the (additive) Jordan-Chevalley decomposition of x, or just the Jordan decomposition; x,, x,, are called (respectively) the semisimple part and the nilpotent part of x.

Proof. Let $a_{1\ldots}$ a_k (with multiplicities $m_1\ldots$ m_k) be the distinct eigenvalues of x, so the characteristic polynomial is ri$(T-a_1)^{m1}$.If V_1=Ker $(x-a_i.1)^{mi}$, then V is the direct sum of the subspaces $V_1\ldots V_k$ each stable under x. Onx clearly has characteristic polynomial

$(T- a_i)m^i$. Now apply the Chinese Remainder Theorem (for the ring F[T]) to locate a polynomial p(T) satisfying the congruences, with pairwise relatively prime moduli: p(T) s al (mod $(T-a_i)^{mi}$), p(T) = 0 (mod T.) (Notice that the last congruence is superfluous if 0 is an eigenvalue of x, while otherwise T is relatively prime to the other moduli.) Set q(T) = T - p(T). Evidently each of p(T), q(T) has zero constant term, since p(T) 0 (mod T).

Set x, = p(x), x,, = q(x). Since they are polynomials in x, xs and x,, commute with each other, as well as with all endomorphisms which commute with x. They also stabilize all subspaces of V stabilized by x, in particular the V,, The congruence p(T) = a; (mod (T-ai)"') shows that the restriction of x.-a• I to Vi is zero for all i, hence that x, acts diagonally on V; with single eigenvalue al. By definition, x,, = x-xs, which makes it clear that x_n is nilpotent. Because p(T), q(T) have no constant term, (c) is also obvious at this point.

It remains only to prove the uniqueness assertion in (a). Let x = s + n ,be another such decomposition, so we have x_s-s = n-x_n Because of (b), all endomorphisms in sight commute. Sums of commuting semisimple (resp. nilpotent) endomorphisms are again semisimple (resp. nilpotent), whereas only 0 can be both semisimple and nilpotent. This forces s = x_s n = x_n.

To indicate why the Jordan decomposition will be a valuable tool, we look at a special case. Consider the adjoint representation of the Lie algebra gI(V), V finite dimensional. If x ∈ gI(V) is nilpotent, then so is ad x (Lemma 3.2). Similarly, if x is semisimple, then so is ad x. We verify this as follows. Choose a basis ($v_i\ldots$ v_n.) of V relative to which x has matrix diag $(a_1\ldots a_{n)}$

Let {e_{ij}} be the standard basis of gl(V) (1.2) relative to (v_1 … v_n): $e_{ij}(v_k)$ = $8_{.jk}v_i$. Then a quick calculation (see formula (*) in (1.2)) shows that ad x (e_{ij}) = (a_i- a_j)e_{ij}. So ad x has diagonal matrix, relative to the chosen basis of gl(V).

Lemma A. Let x ∈ End V (dim V < oo), x = x_s+x_n its Jordan decomposition. Then ad x = ad x_s+ad x_s is the Jordan decomposition of ad x (in End (End V)).

Proof We have seen that ad x_s ad x_n are respectively semisimple, nilpotent; they commute,since [adx_s adx_n.] = ad [x_s x_n]=0. Then part (a) of the proposition applies.
A further useful fact is the following.

Lemma B. Let V be a finite dimensional F-algebra. Then Der V contains the semisimple and nilpoten: parts (in End V) of all its elements.

Proof If S ∈ Der V, let a, v E End 91 be its semisimple and nilpotent parts, respectively. It will be enough to show that a ∈ Der V. If a ∈ F, set V_a = {x ∈ v| a.1)^k x = 0 for some

k (depending on x)}. Then V is the direct sum of those V,, for which a is an eigenvalue of 8 (or a), and a acts on $V^a V_b$. as scalar multiplication by a. We can verify, for arbitrary a, b ∈ F, that $V_a.V_b \subset {}_{a+b}$, by means of the general formula: (*) (6 - (a + b).1)"(xy)

4.1. Killing form

4.1.1. Criterion for semisimplicity

Let L be any Lie algebra . if x,y Є L define κ (x, y) = Tr (ad x ad y).Then κ is a symmetric bilinear form on L , called the **Killing form** . κ is also **associative** , inthe sense that
κ ([x, y], z) = κ (x, [y, z]) .
Tr ([x, y], z) = Tr (x, [y, z]) for endomorphisms , y , z of finite dimensional vector space.

Lemma 4.1.1.1
Let I be an ideal of L . If κ is the Killing form of L and κ_1 the killing form of I (viewed as Lie algebra), then $\kappa_I = \kappa|_{I\times I}$.

Proof
First , a simple fact from linear algebra : If W is a subspace of a finite dimensional vector space V , and ф an endomorphism of V mapping V into W, then Tr ф = Tr (ф|W). Now if x, Є I , then (ad x)(ad y) is an endomorphism of L ,mapping L into I , so its trace κ(x, y) coincides with the trace κ_I (x, y) of
(ad x)(ad y)$|_I$ = (adI)(ad $_I$ y).
Hence the proof.

Definition 4.1.1.1

A symmetric bilinear form β(x, y) is called **non degenerate** if its **radical** S is 0 where S = {x Є L / β(x, y) = 0 ∀ y Є L }.Because the Killing form is associative , its radical is more than just a subspace : S is an *ideal* of L.

Fix a basis x1,...,n of L . Then κ is non degenerate iff the $n \times n$ matrix whose i, entry is κ (x_i , x_j) has nonzero determinant.

Cartan's Criterion

"Let L be a sub algebra of (V), V finite dimensional. Suppose that

Tr (xy) = 0 ∀ x Є [L, L] , y Є L . Then L is solvable ".

Proof. As remarked at the beginning of (4.3), it will suffice to prove that *[LL]* is nilpotent, or just that all x in *ILL]* are nilpotent endomorphisms (Lemma 3.2 and Engel's Theorem). For this we apply the above lemma to the situation : *V* as given, *A = [LL],* ***B*** *= L,* so *M = {x* Є g1(V)I[x, *L]* **⊂** *[LL] }.* Obviously *L* ⊂ *M.* Our hypothesis is that *Tr(xy)* = 0 for x Є *[LL],* y Є *L,* whereas to conclude from the lemma that each x Є *[LL]* is nilpotent we need the stronger statement: *Tr(xy)* = 0 for *x* Є *[LL]*, *y* Є *M*.

Now if *[x, y]* is a typical generator of *[LL]*, and if z **e** *M*, then identity (*) above shows that *Tr([x, y]z) = Tr(x[y, z]) = Tr([y, z]x).* By definition of *M,* [y, ***z]*** **e** *[LL],* so the right side is 0 by hypothesis.

Theorem 4.1.1.2

Let L be a Lie algebra . Then L is semisimple iff its Killing form is non degenerate.

Proof

Suppose first that Rad L = 0. Let S be the radical of κ. By definition,

Tr (ad x ad y) = 0 ∀ x Є S , y Є L . According to *Cartan's criterion*, ad_L S is solvable, hence S is solvable. Since S is an ideal of L ,

so S ⊂ Rad L = 0 ,and κ is non degenerate.

Conversely, let S = 0. To prove that L is semisimple, it will suffice to prove that every abelian ideal I of L is induced in S . Suppose x Є I , y Є L . Then ad x ad y maps

$L \to L \to I$, and $(\text{ad } x \text{ ad } y)^2$ maps L into [I , I] = 0. This means that

ad x ad y is nilpotent, hence that

0 = Tr (ad x ad) = κ (x, y), so $I \subset S = 0$.

4.1.2 Simple ideals of L

Definition 4.1.2.1

A Lie algebra L is said to be the **direct sum** of ideals $I_1,...,I_t$ provided L = I 1+...+I t (direct sum of subspaces). This condition forces

$[I_i, I_j] \subset I_i \cap I_j = 0$ if $i \neq$. We write $L = I_1 \oplus ... \oplus I_t$.

Theorem 4.1.2.1

Let L be semisimple. Then there exist ideals L 1,...,L t of L which are simple (as Lie algebras), such that $L = L_1 \oplus ... \oplus L_t$. Every simple ideals of L coincides with one of the L_i. Moreover, the Killing form of L_i is the restriction of κ to $L_i \times L_i$.

Proof

As a first step, let I be an arbitrary ideal of L . Then

$I^{\perp} = \{x \in L \mid \kappa(x, y) = 0 \ \forall\, y \in I\}$ is also an ideal, by the associativity of κ.

Cartan's Criterion, applied to the Lie algebra I , shows that the ideal $I \cap I^{\perp}$ of L is solvable (hence 0). Therefore, since dim I +dim $I^{\perp}$= dim L , we must have $L = I \oplus I^{\perp}$.

Now proceed by induction on dim L to obtain the desired decomposition into direct sum of simple ideals. If L has no nonzero proper ideals, then L is simple already and we are done. Otherwise let L_1 be a minimal nonzero ideal ; by the preceding paragraph, $L = L_1 \oplus L_1^{\perp}$. In particular, any ideal of L_1 is also an ideal of L, so L_1 is semisimple. For the same reason, $L_1^{\perp}$ is semisimple; by induction, it splits into a direct sum of simple ideals, which are also ideals of L.

The decomposition of L follows.

Next we have to prove that these simple ideals are unique. If I is any simple ideal of L , then [I, L] is also an ideal of I , nonzero because Z (L) = 0; this forces [I, L] = I . On the other hand, $[I, L] = [I, L_1] \oplus ... \oplus [I, L_t]$, so all but one summand must be 0 . Say

[I, L_i = I. Then I ⊂ L_i, and I = L i (because L_i is simple). The last assertion of the theorem follows from lemma 4.1.1.1.
Hence the proof.

4.1.3. Inner derivations

Observation : (*) [δ, ad x] = ad (δx) , x ∈ L , δ ∈ Der L .

Theorem 4.1.3.1
If L is semisimple , then ad L = Der L (i.e., every derivation of L is inner).

Proof
Since L is semisimple, Z (L) = 0 . Therefore, L → ad L is an isomorphism of Lie algebras. In particular, M = ad L itself has non degenerate Killing form. If D = Der L , we just remarked that [D, M] ⊂ M . This implies that κM is the restriction to M × M of the Killing form κ_D of D. In particular, if I = M $^{\perp}$ is the subspace of D orthogonal to M under κD, then the non degeneracy of κM forces I ∩ M = 0. Both I and M are ideals of D , so we obtain
[I, M] = 0 .If δ ∈ I , this forces ad (δx) = 0 ∀ x ∈ L (by (*)), so in turn δx = 0
(x ∈ L) because ad is one to one and δ = 0.
Conclusion: I = 0, Der L = M = ad L .
Hence the proof.

4.1.4. Abstract Jordan decomposition

Proposition 4.1.4.1
Let V be a finite dimensional vector space over F, x ∈ End V.
a) There exist unique x_s, x_n ∈ EndV satisfying the conditions: x= x_s+x_n, x_s is semisimple, x_n is nipotent ,$_s$ and x_n commute.
b) There exist polynomials p(T) , q(T) in one indeterminate , without constant term,
x_s = p(x) , x_n = q(x). In particular, x_s and x_n commute with any endomorphism commuting with x.
c) If A ⊂ B ⊂ V are subspaces and x maps B into A, then x_s and x_n also map B into A.
The decomposition x = x_s + x_n is called the **Jordan – Chevalley decomposition**

of x or just the **Jordan decomposition**; x_s,x_n are called(respectively) the semisimple part and nilpotent part of x.

Lemma 4.1.4.2
Let U be a finite dimensional F-algebra. Then Der U contains the semisimple and nilpotent parts(in End U) of all its elements.

Abstract Jordan decomposition
In particular, since Der L coincides with ad L while L → ad L is one to one each x Є L determines unique elements s, Є L such that ad x = ad s + ad n is the usual Jordan decomposition of ad x (in End L). This means that x = s + n,
with [s, n] = 0, s ad-semisimple(i.e., ad S semisimple), n ad-nilpotent.

We write s = x_s , n = x_n , and call these the semisimple and nilpotent parts of x.
The abstract decomposition of x just obtained does in fact agree with the usual Jordan decomposition in all such cases.

4.2. Complete reducibility of reprsentations

4.2.1. Modules

A vector space V, endowed with an operation L ×V →V (denoted (x, v) ↦ $x.v$ or just) is called an L-**module** if the following conditions are satisfied.
M1: (ax+ by).v = a($x.v$) + b($y.v$)
M2: x.(av + bw) = a ($x.v$) + b($x.w$)
M3: [x, y].v = $x.y.v$ – $y.x.v$
(,y Є L ; v,w Є V ; a,b Є F)
For example, if ф: L → (V) is a representation of L, then V may be viewed as an L - module via the action $x.v$ = ф(x) (v).
Conversely, given an L -module V, this equation defines a representation
ф: L → (V) .
A **homomorphism of** L- **modules** is a linear map ф: V →W such that ф($x.v$) = x.ф(v).
The kernel of such a homomorphism is then an L-submodule of V. When ф is an isomorphism of vector spaces, we call it an **isomorphism** of L -modules; in this case the

two modules are said to afford **equivalent** representation of L. An L -module V is called **irreducible** if it has precisely two L -submodules.

V is called **completely reducible** if V is a direct sum of irreducible L -submodules.

Lemma 4.2.1.1 (Schur's Lemma)

Let ϕ: L → (V) be irreducible. Then the only endomorphisms of V commuting with all ϕ(x) (x Є L) are the scalars.

4.2.2. Casimir element of a representation

Let L be semisimple and let ϕ: L →(V) be faithful (i.e., one to one) representation of L . Define a symmetric bilinear form β(x, y) = Tr (ϕ(x) ϕ(y)) on L . The form β is

associative, so in particular its radical S is an ideal of L. Moreover β is non degenerate and by *Cartan's Criterion* we have ϕ(S) ≅ S is solvable. So, S = 0. Now let L be semisimple, β any non degenerate symmetric associative bilinear form on L. If ($x_{1,2,...,}x_n$) is a basis of L , there is a uniquely determined dual basis (y_1,y_2,...,y_n) relative to β, satisfying

β(xi,yj) = δij . If x Є L , we can write

[x, xi] = Σj a ij xj and [x, yi] = Σj bij yj.

Using the associativity of β, we compute

aik = Σj aij β(xj,yk) = β([x, xi],yk)

= β(-[xi ,x], yk) = β(xi,-[x, yk])

= -Σj bkjβ(xi,yj)

= - bki

If ϕ:L → gI(V) is any representation of L , write c ϕ(β) = Σi $\phi(xi)\phi(yi)$ Є End V.

Using the identity (in End V), [x, yz] = [x, y] + [x, z] and the fact that aik = - bki, we obtain:

[ϕ(x), c ϕ(β)] = Σ[$\phi(x)$, $\phi(xi)$] $\phi(yi)$ + Σi $\phi(xi)$ [$\phi(x)$, $\phi(yi)$]

= Σi, a $i(xj)$ $\phi(yi)$ + Σi,j bij ϕ (xi) $\phi(yj)$ = 0 .

i.e., c ϕ(β) is an endomorphism of V commuting with ϕ(L). We can conclude that,

Let ϕ: L → (V) be a faithful representation with trace form β(x, y) = Tr (ϕ(x) ϕ(y)). In this case, having fixed a basis (x1,2,...,xn) of L, we write simply c ϕ for c ϕ(β) and call this the **Casimir element of ϕ** .

Its trace is

Σ i ((xi) ϕ(yi)) = Σi β(xi,yi) = dim L.

CONCLUSION

Lie Algebra arises naturally in the study of mathematical called Lie groups which serves as groups of Transformations on spaces with certain symmetries. Also introduced the basic concepts and some algebraic facts related to Lie Algebra. In this project we gave a collection of typical examples of Lie Algebra and introduced the basic vocabulary of Lie Algebra. We discussed solvability, nilpotency and semi simple Lie Algebras which plays an important role in the study of Lie Algebra.

Lie Algebra plays a fundamental role in modern mathematical physics. When focus on the recent advances in the applications of Lie Algebra, we can see it covered a wide areas of topics in interdisciplinary studies in mathematics, mechanics, physics and finance. Based on the linear structure of Lie Algebra, many statistical learning methods can be readily applied.

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