Riemann Surfaces and Mappings on it

Project Report Submitted to

Mahatma Gandhi University

In partial fulfilment of the requirement for the award of the degree

MASTER OF SCIENCE IN MATHEMATICS

Submitted By

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2020-2022

ACKNOWLEDGEMENT

With great pleasure I express my sincere gratitude to each and every one who helped me for the successful completion of this project. I would like to acknowledge and give my warmest thanks to Dr. Lakshmi C, Department of Mathematics, Bharata Mata College, Thrikkakara. Her guidance and advice carried me through all the stages of writing my project.

I heartily thank all other teachers of our department for their constant support and guidance. I would also like to give special thanks to my parents and friends for their continuous support and understanding when undertaking my project. Above all, I thank God Almighty for letting me through all the difficulties and guiding me to successfully complete my project.

. **DONIYA LAIJU**

Place: Thrikkakara

Date: 25-09-2022

DEPARTMENT OF MATHEMATICS

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THRIKKAKARA, KOCHI-682021

CERTIFICATE

This is to certify that the project on " RIEMANN SURFACES AND MAPPING S ON IT " is a bonafide work done by Ms. DONIYA LAIJU carried out here in the Department of Mathematics, Bharata Mata College, Thrikkakara, under my supervision, in partial fulfilment of the requirements of the degree of Master of Science in Mathematics of Bharata Mata College, Thrikkakara.

Dr Lakshmi C
(Project Guide)

Dr. Seethu Varghese
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<u>DECLARATION</u>

I hereby declare that the project work entitled "RIEMANN SURFACES AND MAPPINGS ON IT" submitted to Mahatma Gandhi University, is a record of an original work done by me under the guidance of Dr. Lakshmi C, Department of Mathematics, Bharata Mata College and this project work is submitted in the partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics. The results embodied in this project have not been submitted to any other University or Institute for the award of any degree or diploma.

DONIYA LAIJU

Place : Thrikkakara
Date : 25-09-2022

CONTENTS

ABSTRACT

In first Chapter we discuss about the required topics to study if a surface is Riemann surface. These include the study of connected space, second countable space, Hausdorff space. And also, we will come across the idea of complex structure and the basic properties constituting complex structure.

In the second chapter we will be discussing in detail about the Riemann surfaces,its peculiarities.Then we will take an overview on the geometry of these surfaces.for this we have to understand basic idea of manifolds.Hence the same is discussed in this chapter .We will look into the compactness of these surfaces too.

In the next chapter we will look into some example of Riemann surface in detail.we will discuss how the charts are formed there,how will the atlases be formed.
And also we will see how they stand as a Riemann surface. In the fourth chapter we will study on holomorphic mappings on these surfaces and how these holds on various Riemann Surfaces. Meromorphic functions, C^{∞} Functions and Harmonic functions which are defined on the Riemann Surfaces are discussed on the final Chapter.

INTRODUCTION

In Mathematics,especially in complex analysis, we deal with some surfaces called Riemann Surfaces, that are complex manifolds of dimension one .These surfaces were first studied by Bernhard Riemann and hence was named after him.To tell more about these surfaces they are some deformed format of complex plane.For some surfaces like Complex Tori,Riemann sphere the complex analysis of these can be done only after introducing a mapping from the points on these surface to the complex plane.This kind of mapping is done so that these surfaces can be shifted to complex plane as a whole sheet.That is, locally these are some points on a complex plane but taking an overview topologically they are some structures in 3 dimension.However merely defining these mappings is not enough.Beside the possibility of being mapped to complex plane they should satisfy some additional requirements that are to be discussed in this.

Throughout this project we will be discussing how these surfaces take on some functions.We can define some mappings called holomorphic mappings between these surfaces.They are helpful in obtaining general behavior of transportation of these surfaces.There are some peculiarities that Riemann Surfaces hold under these mappings.
Besides holomorphic mappings we will be looking more into meromorphic functions.Also we look how meromorphic functions are found on certain Riemann surface.Then we will just take a quick glance on Harmonic functions and $C\infty$ functions.

Application Riemann surface is that it helps in determination of several holomorphic functions.Especially those functions which are multivalued like square root ,logarithm etc.Here it act as platform for observing global behaviour of these functions.

This work is an attempt to study about Riemann Surfaces along with its examples.Also through this project we will take a little look on how holomorphic mappings work on these surfaces and also about meromorphic functions.These functions act as fundamentals for analysing many other property of Riemann surface.

1.PRELIMINARIES

A topological space should satisfy certain conditions for it to be Riemann Surface.This chapter deals with the acquaintance to those basic conditions.

Definition 1.1

Connected space

A topological space is connected if it cannot be represented as union of two or more disjoint non empty open subsets,that is, if X is a topological space,then X is connected if it is impossible to find non empty subsets A and B of it such that $X = A \cup B$ and $\bar{A} \cap \bar{B} = \Phi$

Definition 1.2

Second Countable space

A topological space is said to be second countable if it has a countable base.

Definition 1.3

Hausdroff space

A topological space X is said to be Hausdroff if for any two distinct points x and y inX there exist two disjoint subsets U and V such that $x \in U$ and $y \in V$.

Now what left to be understood is about the complex structure.But there is a longpath to get the idea of complex structure .Some basic definitions are to be gone through to understand it those are the following:-

Definition1. 4

Complex Chart

Let X be a space , a complex chart on X is a homeomorphism $\varphi: U \rightarrow V$where U is anopen subset of X and V is an open subset of C **,**the complex plane.

Example 1.5

Taking X as R^2 defining $\varphi : U \rightarrow V$ where $U \subseteq R^2$ and $V \subseteq C$ defined by $\varphi(x,y)= x+iy$ in a complex plane can be regarded as complex chart.

Definition 1.6

Compatible charts

Let φ and ψ be two complex charts on X such that $\varphi : U_1 \rightarrow V_1$ and $\psi : U_2 \rightarrow V_2$ where U_1 and U_2 are open subsets in X and V_1 , V_2 areopen subsets in C**.** Now φ and ψ are said to be compatible if $\psi \circ \varphi^{-1}:\varphi(:U_1 \cap U_2) \rightarrow \psi(:U_1 \cap U_2)$ is holomorphic.

Example 1.7

As in example 1.5 we take $\varphi:U \rightarrow V$ and $\psi:V\rightarrow W$ is a holomorphic bijection between V and W so that φ and $\psi\circ\varphi$ are both compatible charts.

Definition 1.8

Atlas

Consider a space X ,then complex atlas A on X is collection of all complex charts φ_a such that each chart in this collection is compatible to each other.Also all U_a of $\varphi_a : U_a \rightarrow V_a$ covers the space X that is $X = \bigcup U_a$

Definition 1.8

Equivalent Atlas

Let A and B be two atlases.Then they are equivalent if every chart in A is compatible with every chart in B.

Definition 1.9

Complex Structure

The equivalence class of atlases on X forms the Complex Structure

2.RIEMANN SURFACES

Riemann Surfaces are surfaces that locally looks like an open set. For some surfaces like S^2,Complex Tori,the complex analysis at the surface is tedious as it is in 3D,but this can be transported to complex plane. Such surfaces are called Riemann Surfaces.

RIEMANN SURFACES

A space is called Riemann Surface if it is a space that is connected,second countable and Hausdroff ,together with a complex structure.

Let us get deep into determining if a surface is Riemann surface.

- Choose the space X
- Choose the open sets { U_α} of X such that X=⋃ U_α
- Define homeomorphism ϕ_α on these U_α
- Check if they compatible hence form the atlas
- Then create its complex structure
- Check if X is connected
- Check if X is second countable
- Check if X is Hausdroff space

If all these are satisfied then we have the Riemann surface

Example 2.1

Consider the space C .It is topologicaly equivalent to R^2.We define ϕ_α : R^2→C by ϕ_α(x , y)= x+iy where U_α ={(x,y) /x,y ∈ R}. Here ϕ_α is a homeomorphism,hence form complex chart thus constituting complex atlas and therefore the complex structure.

Also C is connected and second countable also Hausdroff, hence we have the complex field as a Riemann surface.

Now we will move onto much more geometrical aspects of these surfaces.
For this we first need to understand what are manifolds.

Definition 2.2

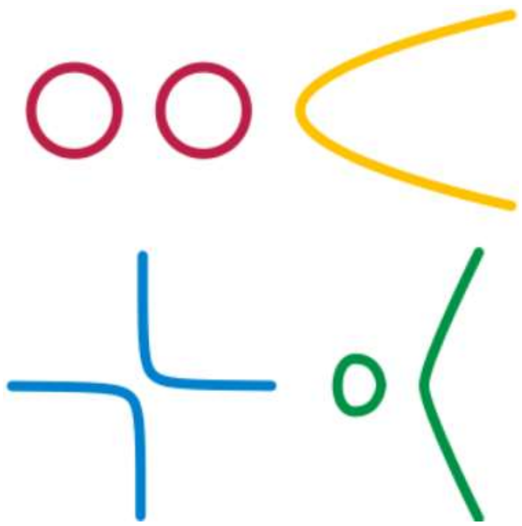
Manifold

A manifold is a topological space that locally is an Euclidean space at each point ,that is,every neighbourhood of point resembles to an Euclidean space.

Definition 2.3

Complex Manifold

A complex manifold is a space in which neighbourhood of every points homeomorphic to an open subset of C^n, and mappings between these open sets are holomorphic functions.



Circles, parabolas, hyperbolas and cubic curves are all 1D Manifolds.

Here we have Riemann surfaces as complex manifold of dimension one.

Definition 2.4

Real-2-manifold

Manifold are defined on the basis of charts and atlases.Here too we are required with these ideas to understand manifolds.A real chart on X is homeomorphism **Φ**:U→V where U ⊂X and V⊂R^n,V is open.Two such real charts ϕ_1 and ϕ_2 are C∞-compatible if $U_1 \cap U_2$ =ϕ or $\phi_2 \circ \phi_1{}^{-1}$ is C∞ diffeomorphism.These real charts aggregate to form C∞ atlas which are collection of pairwise C∞ -compatible charts.Thus we have equivalent atlases ,that form C∞ structure.

Here Riemann surfaces are generally told as an example of real 2 manifold .

Thus we can conclude that ,generally,every Riemann surfaces are complex one manifolds or can be interpreted as real two manifolds.

Proposition 2.1

Every Riemann surface is an orientable path connected C∞ real manifold of dimension 2.

Every Riemann surface is a one dimensional manifold in C then it is connected and thus path connected.

Also mapping we defined through charts are holomorphic especially between two complex subsets.then it preserves orientation ,the angle between mapping etc.Therefore the Riemann surface is orientable.Now to know about C∞ structure we have the real charts ϕ_i:U_i→$V_i$where U_i ⊆X and V_i ⊆R^n.Now C∞ Compatible charts areϕ_1 and ϕ_2 such that $\phi_2 \circ \phi_1{}^{-1}$:ϕ_1($U_1 \cap U_2$) →ϕ_2($U_1 \cap U_2$) is a C∞ diffeomorphism.
(diffeomorphism-the mapping and its inverse have partial derivatives of all order)

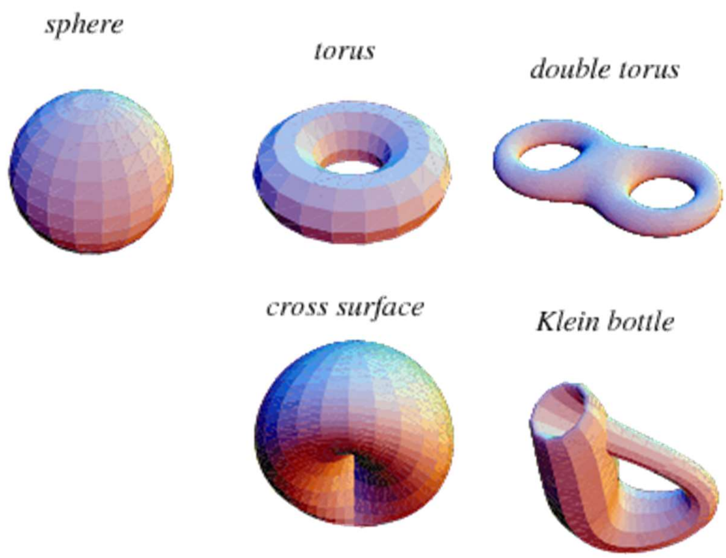
The charts mentioned above form atlases and its equivalence class form C^∞ structure.but C^∞ real manifold is the topological space with C^∞ structure that is hausdroff and second countable

Proposition 2.2

Consider a g-holed torus,then the Riemann surfaces that are compact will homeomorphic to these.

Definition 9

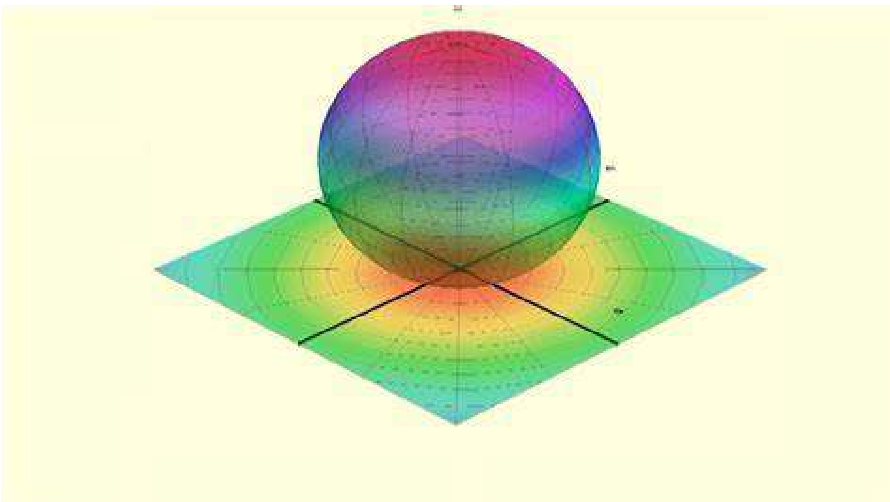
The compact orientable real two manifolds can be classified into g-holed torus for $g \geq 0$. If g=0 we have no holes then it is a sphere,if g=1 it is simple torus and has one hole. This g is called topological genus.
Such surfaces can be obtained even if $g \geq 2$ but this g should be unique.



Examples of 2D manifolds such as a sphere, torus, double torus, cross surfaces and Klein bottle

Thus Riemann surfaces holds structures diffeomorphic to these.

Thus through this chapter we have analysed the geometrical idea of Riemann surface. For each point taken on the Riemann surface we have a ball of small radius with that point as centre that can be mapped to a ball of same radius with particular centre on the complex plane.now integrating all these points on complex plane that is image of those points on the surface we will be getting a thin sheet in complex plane.thus it can be concluded that Riemann surfaces can be considered as this sheet in the complex plane making its complex study possible.



Sphere which is a riemann surface been considered as complex plane

3.EXAMPLES OF RIEMANN SURFACES

Now lets look into several examples of Riemann surfaces in detail.

Example 1

Riemann Sphere (C∪∞)

Consider S^2 in R^3, S^2= {(a, b, c)∈ R^3 $/a^2 + b^2 + c^2 = 1$}

Now let c=0 be the plane that is akin to complex plane.now for this plane we have (a,b,0) as the points and these represents z=a+ib on the complex plane.

Consider a point (0,0,1) on S^2 then we define ϕ_1:S^2-(0,0,1)→C such that ϕ_1(a,b,c)=(a⁄1-c)+i(b⁄1-c)

Now for point (0,0,-1) on S^2 we have ϕ_2:S^2-(0,0,-1)→C such that ϕ_2 (a,b,c)=(a⁄1-c)-i(b⁄1-c)

Now $\phi_1{}^{-1}(z)$=(2Rez⁄ ($|z|^2 + 1$) ,2 imz⁄ ($|z|^2$+1), ($|z|^2 - 1$)/($|z|^2$+ 1))

Similarly,

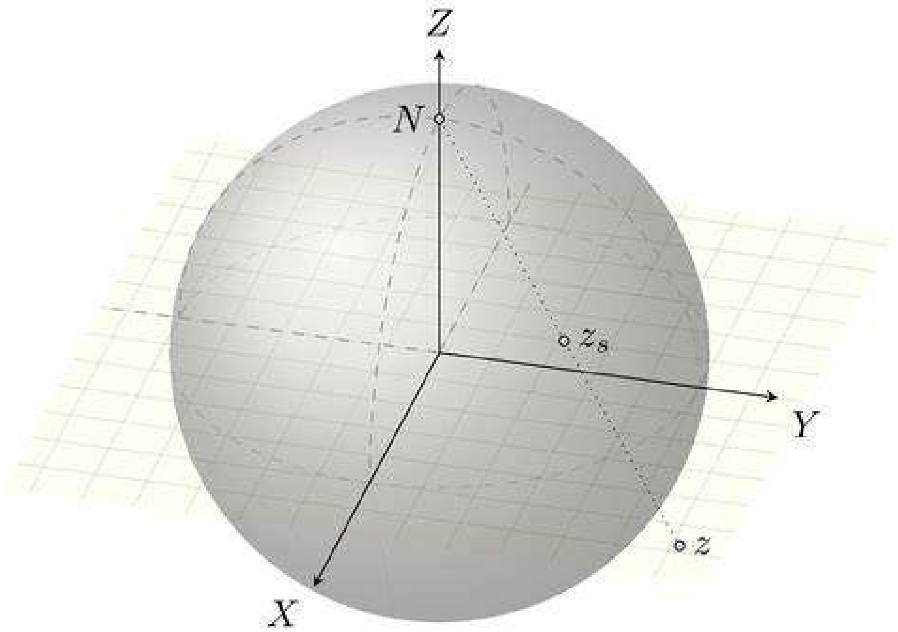
${\phi_2}^{-1}(z)$=(2Rez⁄ ($|z|^2$ + 1) ,-2 imz⁄ ($|z|^2$+1), (1-$|z|^2$)/($|z|^2$+ 1))

Thus for sphere with common domain as S^2-(0,0,∓1).

We have both ϕ_1 andϕ_2 mapped to C* and verify we get

$\phi_2 \circ {\phi_1}^{-1}$ (z)=1/z,thus they are compatible and therefore we will have complex structure with these functions.

Now the sphere is clearly hausdroff and connected and therefore is Riemannn surface.thisRiemann surface is called Riemann surface.



A Riemann sphere

Example 2

Projection Lines,CP^1

Complex projective lines are the subspaces of C^2,if (a,b)≠(0,0) is a vector in C^2 it lies on some projective line and its span is a point on the point CP^1.

Let [a:b] denote the span of (a,b)thus every point on this C$P^1$is of form [a:b],note that a,b≠0 and [a:b]=[ɣa:ɣb] for some ɣ∈C*

Now these arrangements to define the complex structure on CP1

Here we take two sets A and B such that

A={[a:b]/ a≠0} and B={[a:b]/ b≠0}

Now CP^1will be subset of ⋃A and ⋃B .Our aim is to define twocharts that are compatible.let ϕ_1:A→C defined by ϕ1([a:b]) = b/a and let ϕ_2 be defined by ϕ_2 ([a:b])=a/b.since a≠0 and b≠0 in A and B respectively make this possible and mapping will be onto C*.That is ϕ_i(A⋃B)=C*.Also then composition

$\phi_2 \circ {\phi_1}^{-1}$ sends b/a to a/b .These charts are compatible.

Now A and B are open sets ,are connected and A∩B≠0 ,thus A⋃B is connected.Now it remains to show that it is Hausdroff .For that consider p and q such that p∈A-B and q∈B-A.This makes p=[1:0],q=[0:1] only points that can be different A and B.Now they can be seperated by ${\phi_1}^{-1}$(D) and ${\phi_2}^{-1}$ (D),where D is open unit disc in C but it is clear that we can get whole CP^1 by taking union of A and B necessarily as the union of ${\phi_1}^{-1}$ (Dclosure)and ${\phi_2}^{-1}$ (Dclosure) ,where D closure is closed unit disc. hence compact thus make whole CP^1 compact .

Thus projection line we define here is compact Riemann surface.

Example 3

Complex Tori

Consider z_1 and z_2 which are linearly independent on R.

Consider L={m_1 z_1+$m_2 z_2/m_1$,m_2 ∈Z},the lattice which is subgroup of C,then we have the factor group X=C/L and we have projection π:C→X.If we take A an open set of X

then $\boldsymbol{\pi}^{-1}(A)$ is open in C.therefore π is continuous.

Also we have C connected ,then X is connected.

Also π is an open mapping.

Choose some δ such that $|Z|$ >2δ.let d(Z_0,δ) be done with the regard

π:d(Z_0,δ) →π(d)

Now π/d:d →π(d) is onto ,continuous ,open,also one-one.

Now we create charts $\boldsymbol{\psi}_{Z_0}$:π(d)→d as the union of π/d,where d=d(Z_0,δ).therefore Ψ's are complex charts. It remains to show that they are compatible charts.

Consider two points z_1 and z_2,choose Ψ_1:Ψz_1:π(d_{z_1})→$d_{z_1}$and Ψ_2:Ψz_2:π(d_{z_2})→d_{z_2} .let U=π(d_{z_1})∩ $\pi(d_{z_2})$.we have to check if it is holomorphic.If U=ϕ,then done,if U≠ϕ.

Let T(z)=$\Psi_2(\Psi_1{}^{-1}$(z))=Ψ_2(π(z)) for z ∈Ψ_1(U).we have to check if T is holomorphic on U.

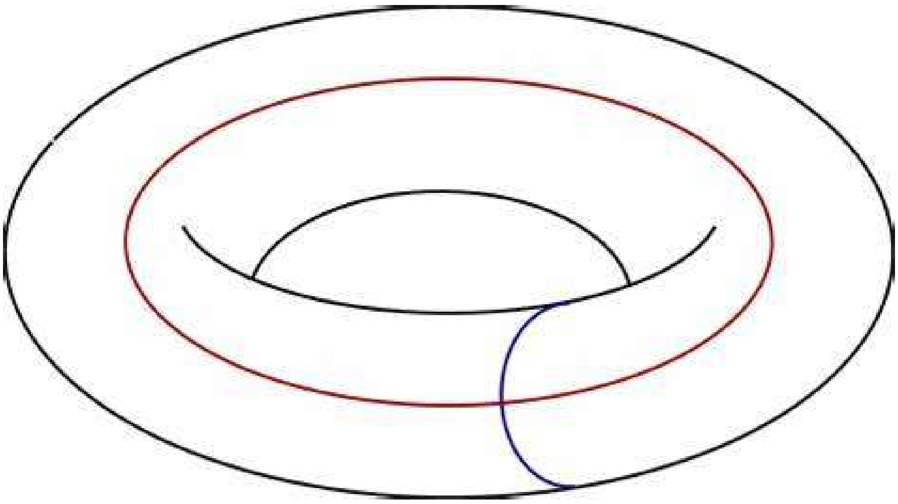
Note that π(T(z))=π(z) for all z ∈Ψ_1(U).so T(z)-z=w(z) ∈L.

Then T(z)=w+z for w ∈L,therefore holomorphic.hence {Ψ_z /z∈C} is complex chart.

Now consider the parallellogram Q_z={z+$\gamma_1 z_1$ +$\gamma_2 z_2$ / γ_i ∈[0,1]}.its is clear that any point on complex plane is congruent modulo L to a point Q_z .therefore Q_z is compact ,so is X.

Here X is a topological space of genus one and X is compact Riemann surface ,so called Complex torus.

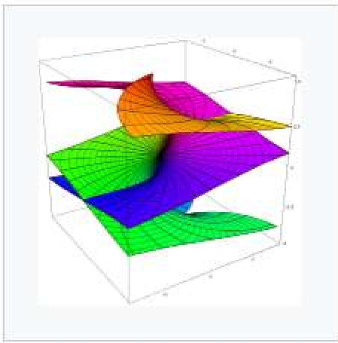
These Riemann surfaces are complex tori.



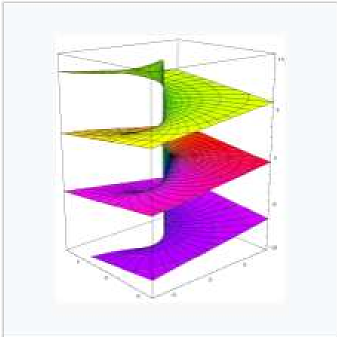
A complex torus of genus one

Beside these there exist many more examples on riemann surfaces like projective plane,lines ,conic etc.

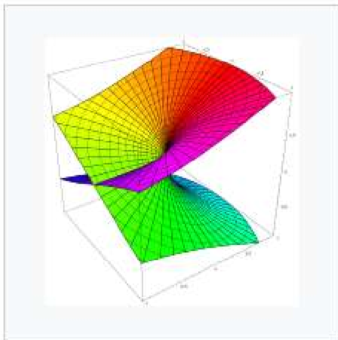
Some of the examples of non-compact Riemann Surfaces is shown below



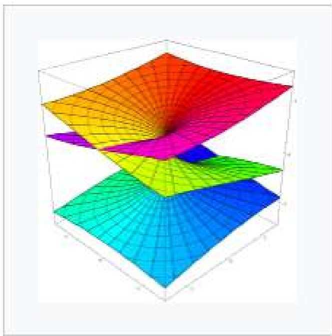
$f(z) = \arcsin z$



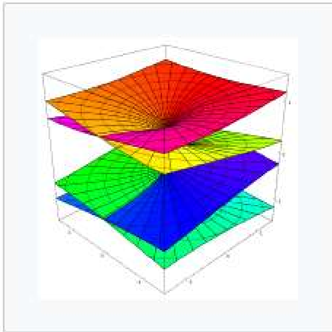
$f(z) = \log z$



$f(z) = z^{1/2}$



$f(z) = z^{1/3}$



$f(z) = z^{1/4}$

4.HOLOMORPHIC MAPS AND RIEMANN SURFACES.

In the chapter 2, we discussed transporting point from a Surface on to the complex plane.we check if f has any particular property at any point .The complex charts are useful in transporting this point to the neighbourhood of a point in a complex plane.so study of these surfaces are done by holomorphic mappings on it where the charts play cruicial role in defining it.

consider a Riemann surface X, let x be any point on it and we define h at neighbourhood of x .suppose we want to check whether h holds a particular property at this neighbourhood,what we will do is just transport the function to complex plane as we were discussing earlier ,so that it is enough to check if the propertty holds in neighbourhood of the corresponding point of x in the plane.

Definition 4.1

Holomorphic functions

Let X be Riemann surface.Let x be a point on it ,let h be complex valued function defined in neighbourhood N of x.

We say that h is holomorphic at x if there exist a chart Ψ:U→V with u∈U such that $h \circ {\Psi_1}^{-1}$ is holomorphic at Ψ(x).

Lemma 4.2

Let X be a Riemann surface,x be a point on it and h be complex valued function defined on N of x,then h is holomorphic at x if and only if for every chart Φ:U→V with x∈U, we have $h \circ \Phi^{-1}$ is holomorphic at Φ(x).

Proof:Consider 2 charts Φ_1 and Φ_2 such that it has x in its domains,suppose that $h \circ {\Phi_1}^{-1}$ is holomorphic at Φ_1 (x) then $h \circ {\Phi_2}^{-1}$ is holomorphic at Φ_2(x), since $h \circ {\Phi_2}^{-1}$=($h \circ {\Phi_1}^{-1}$) ∘($\Phi_1 \circ {\Phi_2}^{-1}$) ,where it is the composition of two holomorphic functions.

If h is holomorphic at q then h is holomorphic at neighbourhood of q.

Also h is holomorphic in N if and only if there exist some Φ_i's such that h∘ ${\Phi_i}^{-1}$ is holomorphic on Φ_i (N∪Ui) for each i ,where Ui is domain of Φ_i.

Now lets look under what conditions does the Riemann surface we met earlier will have holomorphic mappings.

Example 4.3

Any complex chart is holomorphic on its domain since it is considered as a complex valued function.

Example 4.4

Let h be a complex valued function defined on an open set in C, we know that C is

Riemann surface and hence by definition itself we have h as the holomorphic.

Example 4.5

Suppose h and f are two holomorphic functions at x∈X ,then h∓f ,hf are holomorphic at x,and h/f is holomorphic if f(x)≠0.

Example 4.6

Let h be a complex valued function defined on Riemann sphere C∞ defined in the neighbourhood of ∞.Then h is holomorphic at ∞ if and only if h(1/x) is holomorphic at point x=0.if we have h(z)=f(z)/g(z) , then h is holomorphic at ∞ if and only if deg (f)≤deg(g).

Example 4.7

For projective line we came across earlier ,let f(z,w) and g(z,w) be two polynomials of same degree then h([z:w])=f(z,w)/g(z,w) is a holomorphic function at [z_0,w_0], provided g(z_0,w_0)≠0

Example 4.8

Consider a complex torus C/L with quotient map $\boldsymbol{\pi}$:C→C/L then
h:N→C be a complex function on N⊂C/L then h is holomorphic at x∈N if and only if there exist a preimage z of x in C such that h ∘$\boldsymbol{\pi}$ is holomorphic at z.

Definition 4.9

If U⊂X is an open set of Riemann surface X,then Ox(U)={h:U →C/h is holomorphic}.It is a C -algebra.It is often denoted by O(U).

Usually when we have some functions at a point in C ,we can check its analyticity .If not ,we can find its singularity and distinguish what type of singularity does it have

.Similarily on Rieman surfaces these can be checked by checking it on the points on C where it was being transferred .

Definition 4.10

Let h be a holomorphic mapping in punctured neighbourhood q ∈X,where X is the Riemann surface.Then

a) h has a removable singularity at q∈X if there exist a chart ,Φ:U→V with q∈U ,such that $h \circ \Phi^{-1}$ have removable singularity at ,Φ(q)

b) h has a pole at q if and only if there exist a chart Φ:U→V with q∈U such that $h \circ \Phi^{-1}$ has pole at Φ(q)

c)h has an essential singularity at q if there exist essential singularity for $h \circ \Phi^{-1}$ at Φ(q) where Φ is defined earlier

That is to check if h has particular singularity at q it is enough to check whether $h \circ \Phi^{-1}$ has that singularity at Φ(q)

- If h(x) is bounded in the neighbourhood of q,then h has removable singularity at q
- if x approaches q and h(x) = ∞ then h has a pole
- If h(x) has no limit as x approaches q,then h has no essential singularity at q.

Theorem 4.11

Let X be compact Riemann surface ,suppose h is holomorphic on all of X.Then h is constant function

to prove this theorem we need to know maximum modulus theorem

Theorem 4.12

The maximum modulus theorem

let h be holomorphic on connected open set N of Riemann surface X ,suppose there is q ∈N such that$|h(x)| \leq |h(q)|$ for all x∈N, then h is constant on N.

proof of theorem 4.11

Since h is holomorphic ,we have h is a continuous function.since X is compact h achieves ,its maximum at some point on X.then theorem 4.12 suggests it is a constant function

Geometrically more idea about the surfaces to study are brought up by defining functions between objects.Here we have the objects as two Riemann surfaces.

Definition 4.13

A mapping F:X→Y is holomorphic at some u∈X if and only if there exists charts $\Psi_{1:}: U_1 \rightarrow V_1$on X with u∈ U_1 and $\Psi_{2:}: U_2 \rightarrow V_2$ on Y with F(u)∈U_2 such that $\Psi_2 \circ F \circ {\Psi_1}^{-1}$ is holomorphic at$\Psi_{1:}$(u).

If F is defined on open set A⊆X ,F is holomorphic on A if all F is holomorphic at each point in A .That is F is holomorphic if and only if holomorphic on all of X.

Example 4.14

Consider I:X→Y,then this map is holomorphic map even if X is taken as any Riemann surface.

Lemma 4.15

Let F:X→Y be mapping between Riemann surfaces,then F is holomorphic on openset A if and only if we can have two collection of charts$\{\Psi_{1i}: U_1 \rightarrow V_1\}$ on X with A⊂$\cup_i U_{1i}$ and $\{\Psi_{2k}: U_2 \rightarrow V_2\}$ on Y with F(A)⊂$\cup_k U_{2k}$ such that Ψ_{2k} ∘F∘ ${\Psi_{1i}}^{-1}$ is holomorphic for every i and j where it is defined.

Example 4.16

If Y is the complex plane C, X be any Riemann surface then F:X→Y is a holomorphic function on X.this is possible since Y is Riemann surface

Lemma 4.17

If F is holomorphic then F is continuous

Lemma 4.18

Let F and G be two holomorphic maps where F:X→Y And G :Y→Z then

G∘F:X→Z is a holomorphic map

Definition 4.19

Isomorphisms and automorphisms

An isomorphism between Riemann surface is a holomorphic mapF:X→Y which is bijective and where inverse F^{-1}:Y→X is holomorphic .when this map is F:X→X is isomorphism ,it is called automorphism on X.

Lemma 4.20

S2 is isomorphic to CP^1

proof: If F: CP^1→S2 in such a way that

F[x_0:x_1]=2Re($x_0\underline{x_1}$),$2Im((x_0\underline{x_1})/(|x_0|^2 - |x_1|^2, |x_0|^2 + |x_1|^2)$

Then this an isomorphism

Theorem 4.21

Open mapping theorem

A non constant holomorphic mapping F:X→Y is open mapping.

Theoram 4.22

F:X→Y be injective holomorphic mapping between two Riemann surface.then F is isomorphisn between X and F(X).

Theorem 4.23

Let X be compact Riemann surface and let F:X→Y be non constant holomorphic map .then Y is compact and F is onto.

Proof:we have F holomorphic and by open mapping theorem F(X) is open,also X is compact.therefore F(X) Is compact.also Y is riemann hence Hausdroff and hence F(X) is closed thus it is clopen on Y and Y is connected.
It must be all of Y .F is onto and Y is compact.

Holomorphic map between two Riemann surfaces has a local normal form.

Proposition 4.24

Let F:X→Y be a nonconstant holomorphic mapping defined at q ∈X .Then we have a integer n≥1 such that for every chart $\Phi_2:U_2 \rightarrow V_2$ on y centered at F(q),there exist charts $\Phi_1:U_1 \rightarrow V_1$ centred at q of X such that $\Phi_2(F(\Phi_1{}^{-1}(z)))$ maps z to z^n

Proof: Fix a chart Φ_2 on Y with centre F(q),choose some other chart Ψ :U→V centred at q of X .Thenwe choose T= $\Phi_2(F(\Psi^{-1}\ (t)))$ and T(z)=$\sum_{i=m}^{\infty} \quad c_i t^i$
with $c_m \neq 0$ and m≥1where T(0)=0
then T(z)=$c_m\ t^m$ +$c_{m+1}t^{m+1}$+$c_{m+2}t^{m+2}$..........
=t^ns(t)
where s(t) is holomorphic at t=0 and s(t)≠0 for t=0.Also here we have some function f(t) that is holomrphic near 0 where $f(t)^n$ =s(t).then T(t)=(tf(t))^n ,
let β (t) = tf(t) = z,here t'(0)≠ 0.Therefore is invertible .Now Φ_1=$\beta \circ \Psi$ is also a chart on X at q.
Therefore $\Phi_2(F(\Phi_1{}^{-1}(z)))$ = $\Phi_2(F(\Psi^{-1}(\beta^{-1}(z)))$
=T(β^{-1}(z))
=T(t)
=(tf(t))^m

$= z^m$

here F maps z to z^m then near q there exist exactly n preimages of points near F(q).

Definition 4.25

Multiplicity

Multiplicity is unique integer n such that F:X→Y maps z near q to z^n

at Y then Multip(F)=n

Example 3.26

Let Φ:U→V be a chart map for X ,consider holomorphic to C .Then has multiplicity one at every point of U.

Thus we can understand properties of surface on which the mappings are done.Holomorphic mappings are those help in converting it into complex plane.

5.MEROMORPHIC FUNCTIONS AND RIEMANN SURFACES

We have dealt with meromorphic functions at some points in complex plane
Here too we are having meromorphic functions defined on these surfaces.
These are done by defining these type of functions on the complex plane where these points are being mapped to

Definition 5.1

Meromorphic functions

A function h is said to be meromorphic at a point qX is either holomorphic ,has removable singularity at q **or** has a pole at q .we say that h is a meromorphic on an open set N if h is meromorphic at every point of N.

Example 5.2

Let h be a complex valued function on some open set in C .Then definition of meromorphic functions are satisfied while considering C as a Riemann surface.

Example 5.3

If h and g are each meromorphic functions at qX,then the product and sum are meromorphic at q moreover h/g is meromorphic at q where g0

Example 5.4

Let h be any complex valued function on Riemann sphere C defined on neighbourhood of ,Then h is meromorphic at if and only if h(1/z) is meromorphic at z=0.Any rational function is meromorphic on all of Riemann sphere.
Suppose h is meromorphic function at some qX ,then it can be regarded as ratio of two holomorphic functions

Example 5.5

Consider the projective line P1 with [z:w].Let f(z,w) and g (z,w) be two homogenous polynomials .Then h([z:w]) =f(z,w)/g(z,w)

Example 5.6

Consider a complex torus C/L we have f:C→L .Let h :N→C be a complex valued function with N the open subset N C/L.Then h is meromorphic at q of N if and only if there exist a image z of q in C such that h is meromorphic at z .as we have seen in chapter two .

Here f=h is always L-periodic ,that is,f(z+w) = f(z) for every wL.there exist injection from functions on C/L and L periodic functions on C.such functions that are periodic are the elliptic functions .

More on meromorphic functions on Riemann surfaces

Let us first look into the order of the meromorphic functions.

Order of meromorphic function at a point

For a meromorphic function ,the order of pole or zero can be determined from the Laurent series expansion of it .So before coming onto its determination lets take a glance on Laurent series

Definition 5.7

Laurent series

Let h be a holomorphic function in a neighbourhood (punctured) of q on X.Let Φ :U→V be a chart on X with q∈U.Let z be a point on X near q, so that z = Φ(x) for x

near q we have $h \circ \Phi^{-1}$ holomorphic at z_0=Φ (q) .therefore we have $h \circ \Phi^{-1}$ as Laurent series expansion about z_0

that is , h($\Phi^{-1}(z)$) =$\sum_i c_i(z - z_0)^i$

This is called Laurent series expansion of h about q with respect to Φ .Here $\{c_i\}$ are the laurent coefficients.

Clearly the type of singularity we are dealing with can be understood from this expansion .If the expansion has no negative terms then it is removable singularity for h.If it is of finite negative terms then it has a pole at q.If the expansion has infinite negative terms then h has essential singularity at h .

Definition 5.8

Order of h at q

Let h be a meromorphic at q,whose Laurent series in local coordinate z is

$\sum_i c_i(z - z_0)^i$

.Then order of h at q, denoted by ord $_q$(h)is the minimum exponent actually in laurent series expansion that is ord $_q$(h) = min {i/c_i ≠0}

Theorem 5.8

Order of h at q is well defined and independent of choice of local coordinates z in the expansion of laurent series.

Proof:

consider another chart Ψ :U1→V1 with q∈U1 with local co-ordinate c=Ψ(x) for x near q .Also Ψ(q) = c_0 .Consider holomorphic map T(c) = $\Phi \circ \Psi^{-1}$ where z=T(c).Any transition function between two compatible charts will not have its derivative as zero in its domain so $T'(c_0)$ ≠0.Then we can create laurent series expansion for z = T(c)=z_0 + $\sum_i a_i(c - c_0)^i$ with $a_0 \neq 0$

Suppose that $b_{i_0}(c - c_0)^i$ +some other terms in laurent series for h at q then $b_{i_0} \neq 0$ so Ordq(h) is i_0.Now for some functions h we can laurent series expansion about c_0 ,we have z-z_0= $\sum_i a_i(c - c_0)^i$ here we see that the least possible ordered term is $b_{i_0} a_1^{i_0}(c - c_0)^{i_0}$ here we cannot have any one of b_{i_0} or $a_1$is zero.So the term exists and hence order of h at q is well defined even if choice of local coordinate is different.

Lemma 5.9

Let h and f be non zero meromorphic functions at q∈ X then

- Ordq(hf) =Ordq(h) +Ordq(f)
- Ordq(h/f) = Ordq(h)-Ordq(f)
- Ordq(1/h)= - Ordq(h)
- Ordq(h±f) min{Ordq(h),Ordq(f)}

Example 5.10

Let h(z) =p(z)/k(z) be a non zero function of z. This can be regarded as a meromorphic function on Riemann sphere.Here we make p and k into linear factors .Then h = c$\prod_i$ (z-λ_i)^{e_i} here c≠0, and λ_i i are distinct complex numbers ,and ei are integers then Ord_{λ_i}(h) = e_i

Ord_{λ_i}(h) = deg (k) - deg (p)

= -$\sum_i \quad e_i$

that is, Ord_x(h) = 0 unless x= ∞ or is x =λ_i

Theorem 5.11

Discreteness of zeroes and poles

Let h be meromorphic function defined on connected open set N of X ,a Riemann surface .if h≠0 identiacally ,then zeroes and poles of this h lies in discrete subset of N.

Theorem 5.12

Suppose h and f are two meromorphic function on connected set N of X,where X is a Riemann surface.Suppose h=f in a subset U of N and has limit in this N.Then h=f on N.

Now let us look into some more functions that can be defined on Riemann surfaces

Definition 5.13

C^∞ Functions

A real valued function z= x+iy is C^∞ at a point z_0 if as a function of x and y ,it has continuous partial derivative of all orders at z_0 .A function h defined on Riemann surface X is C^∞ at a point q if there exist chart Φ:U→V on X with q∈U such that h∘ Φ^{-1} is C^∞ is at Φ(q)

Definition 5.14

Harmonic functions

A real valued C^∞function f(x,y) defined on U⊆ R^2 is harmonic if $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$

identically on U .Now lets transport this property of being harmonic .As usual these are also checked by charts .Suppose f is C^∞ at q on X , f is harmonic at q if there is a chart Φ :U→V with q∈ U such that f∘ Φ^{-1} is harmonic near Φ(q)

Meromorphic functions on Riemann surface

Any rational functions h(z)=f(z)/g(z) is meromorphic on riemann surface on Riemann surfaces . Conversely,meromorphic function of Riemann sphere is rational function.

Theorem 5.15

Any meromorphic function on Riemann surface is rational

Proof:Consider the Riemann sphere C_∞, let h be a function on it. we know that C_∞ is compact Riemann surface.So it will have only finite number of zeroes and poles .Let $\{\lambda_i\}$ be the set of zeroes and poles .Assume that Ord_{λ_i} f=e_i .Consider the function r(z)= $\prod_i$ (z-$\lambda_i)^{e_i}$ which has some zeroes and poles.Let f(z) = h/r(z),f is meromorphic function on C_∞ with no zeroes or poles in the finite plane .therefore as a function on C_∞ ,it is holomorphic and has taylor series

f(z)=$\sum_{n=0}^{\infty} c_n z^n$

this expansion converges on C .Also f is meromorphic at z=∞, in terms of coordinate w=1/z at ∞,we have ,

f(w)= $\sum_{n=0}^{\infty} c_n w^{-n}$

this function is meromorphic at w=0 if f has finitely many terms where it will be a polynomial in z.then we can find a zero in some point in the finite part of C .Therefore we reached contradiction .Hence h/r isi a constant and hence h is rational

Corollary 5.16

Let f be any meromorphic function on Riemann sphere.Then $\sum_p$ Ord_p(f) = 0

Order of meromorphic function h is positive as the zeroes and negative at the poles.Here sum of orders is 0 .it implies that h has same number of zeroes and poles.

CONCLUSION

Through different phase of development we came across fundamental ideas about Riemann Surfaces. We started by introducing the basic requirements forRiemann surfaces,then went through geometrical aspects of these.we analysed some examples of these such as Riemann sphere,the projective line and complex torus.then we viewed what are holomorphic mappings between Riemann surfaces.here we could find that under a holomorphic we had S^2 isomorphic to CP1.Relevance of Riemann surfaces comes when it verify the global behaviour of holomorphic functions on these.Riemann Surface plays vital role in algebraic geometry.

In addition to above mentioned we discussed meromorphic mappings on these surfaces.itholds some special properties on these surfaces.then we analysed laurent series expansion at a point on these surfaces.we went through the idea of harmonic functions on these.also we saw C^{∞} functions.the idea we finally derived is that these functions are defined as same as those on usual complex plane.but the key pointwas that it was the holomorphic functions that kept this function valid.