21001241

21001241

Reg. No..............................

Name..................................

M.Sc. DEGREE (C.S.S.) EXAMINATION, NOVEMBER 2021

Fourth Semester

Faculty of Science

Branch I (A)–Mathematics

MT 04 E02—COMBINATORICS

(2012 to 2018 Admissions—Supplementary/Mercy Chance)

Time : Three Hours | Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. Show that $(4n)!$ is a multiple of $2^{3n} \cdot 3^n$ for each natural number n.
2. Find the number of ways to choose a pair $\{a,b\}$ of distinct numbers from the set $\{1,2,\ldots,50\}$ such that $|a-b|=5$.
3. Define Ramsey's numbers.
4. Show that among any group of 7 people, there must be atleast 4 of the same sex.
5. Show that $\phi(m\,n)=\phi(m)\,\phi(n)$ where $m,n \in \mathrm{N}$ with $(m,n)=1$.
6. Find the number of leap years between 1,000 and 3,000 inclusive.
7. Each of the 3 boys tosses a die once. Find the number of ways for them to get a total of 14.
8. Let (a_r) denote the number of r-permutations of the multi-set $\{\infty \cdot b_1, \infty \cdot b_2, \ldots, \infty \cdot b_k\}$. Find the exponential generating function of (a_r).

(5 × 1 = 5)

Part B

Answer any **five** *questions.*
Each question has weight 2.

9. Prove that $\binom{n}{r}=\binom{n-1}{r-1}+\binom{n-1}{r}$ both algebraically and combinatorially.
10. Let A be a $2n$-element set where $n \geq 1$. Find the number of different pairings of A.

Turn over

11. Show that for any set of 10 points chosen within a square whose sides are of length 3 units, there are two points in the set whose distance apart is atmost $\sqrt{2}$.

12. Let $X \subseteq \{1, 2, \ldots, 99\}$ and $|X| = 10$. Show that it is possible to select two disjoint non-empty proper subsets Y, Z of X such that $\sum(y \mid y \in Y) = \sum(z \mid z \in Z)$.

.

13. Find the number of integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 20, 1 \le x_1 \le 5, 0 \le x_2 \le 7,$ $4 \le x_3 \le 8, 2 \le x_4 \le 6.$ For any q finite sets $A_1, A_2, \ldots, Aq, q \ge 1,$ prove that $\left|A_1 \cup A_2 \quad \cup A_q\right| =$

$$\sum_{i=1}^{q} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| ----+(-1)^{q+1} |A_1 \cap A_2 \ldots \wedge A_q|.$$

14. Find the number of integer solutions to the equation $x_1 + x_2 + x_3 = 28, 3 \le x_1 \le 9, 0 \le x_2 \le 8,$ $7 \le x_3 \le 17$.

15. Prove that for each $u \in \mathrm{N},$ the number of partitions of n into parts each of which appears atmost twice, is equal to the number of partitions of n into parts the sizes of which are not divisible by 3.

16. Solve $a_n = 3a_{n-1} - 2a_{n-2}$ given that $a_0 = 2, a_1 = 3$.

(5 × 2 = 10)

Part C

Answer any **three** *questions.*
Each question has weight 5.

17. Let S be the set of natural numbers whose digits are chosen from $\{1,3,5,7\}$ such that no digits are repeated. Find (i) $|S|$; (ii) $\sum_{n \in S} n.$

18. (a) For all integers $p, q \ge 2$ show that $R(p,q) \le R(p-1,q) + R(p,q-1)$.

    (b) Prove that for all integers $p, q \ge 2$ if $R(p-1,q)$ and $R(p,q-1)$ are even then :

    $R(p,q) \le R(p-1,q) + R(p,q-1) - 1$.

19. Let n and k be positive integers and let S be a set of n points in the plane such that :

   (i) no three points of S are collinear and

   (ii) for any point P of S, there are atleast K points of S equidistant from P. Prove that :

   $$k < \frac{1}{2} + \sqrt{2n}.$$

20. (a) Let $F(n,m), n,m \in N$ denote the number of surjective mappings from N_n to N_m. Show that :

   $$F(n,m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$$

   (b) Show that $\sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n = 0$ if $n < m$.

   (c) Show that $\sum_{k=0}^{n} (-1)^k \binom{m}{k} (n-k)^n = n!$

21. Solve the recurrence relation :

   $a_n - 3a_{n-1} + 2a_{n-2} = 2^n, a_0 = 3, a_1 = 8$.

22. (a) Show that for integers n, r, k such that $n \geq r \geq k \geq 0$ and

   $$r \geq 1 \;\; D(n,r,k) = \frac{\binom{r}{k}}{(n-r)!} \sum_{i=0}^{r-k} (-1)^i \binom{r-k}{i} (n-k-i)! \cdot$$

   (b) Show that for any $n \in N$:

   $$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right).$$

(3 × 5 = 15)