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Reg. No..............................

Name..................................

M.Sc. DEGREE (C.S.S.) EXAMINATION, NOVEMBER 2021

Fourth Semester

Faculty of Science

Branch I (A)–Mathematics

MT 04 E01—ANALYTIC NUMBER THEORY

(2012 to 2018 Admissions—Supplementary/Mercy Chance)

Time : Three Hours | Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. Show that Dirichlet multiplication is associative.

2. Show that if g is multiplicative, then g^{-1} is also multiplicative.

3. Show that $[x]! = \prod_{p\leq x} p^{\alpha(p)}$ for every $x \geq 1$, and $\alpha(p) = \sum_{m=1}^{\infty}\left[\frac{x}{p^m}\right]$.

4. Show that $\psi(x) = \sum_{m \leq \log_2 x} \mathcal{J}\left(x^{1/m}\right)$.

5. Define Chebyshev's ψ and $\mathcal{J}$ functions.

6. Show that if $a \equiv b(\operatorname{mod} m)$, then $(a,m) = (b,m)$.

7. If $(a,b) = d$, show that there exists integers x and y such that $ax + by = d$.

8. Given $m \geq 1, (a,m) = 1$ and $f = \exp_m(a)$. Show that $1, a, a^2, \ldots a^{f-1}$ are incongruent mod m.

(5 × 1 = 5)

Part B

Answer any **five** *questions.*
Each question has weight 2.

9. Define the Mangoldt function $\wedge(n)$. Show that $\log n = \sum_{d/n} \wedge(d)$.

10. Show that if both g and $f * g$ are multiplicative, then f is also multiplicative.

Turn over

11. Show that if $x \geq 1$ and $\alpha > 0, \alpha \neq 1$ $\sum_{n \leq x} \sigma_\alpha(n) = \frac{\xi(\alpha+1)}{\alpha+1} x^{\alpha+1} + O(x^\beta)$, where $\beta = \max\{1, \alpha\}$.

12. Show that there is a constant A such that $\sum_{p \leq x} \frac{1}{p} = \log \log x + A + O\left(\frac{1}{\log x}\right)$ for all $x \geq 2$.

13. Show that $a = b(\bmod m)$ if and only if a and b give the same reminder when divided by m.

14. State and prove Wolstenholme's theorem.

15. State and prove Chinese reminder theorem.

16. Let p be an odd prime and let d be any positive divisor of $p - 1$. Prove that in every reduced residue system mod p there are exactly $\phi(d)$ numbers a such that $\exp_p(a) = d$.

(5 × 2 = 10)

Part C

Answer any **three** *questions.*
Each question has weight 5.

17. (a) If $x \geq 2$ show that $\log[x]! = x \log x - x + O(\log x)$.

(b) For $x \geq 2$ show that $\sum_{p \leq x} \left[\frac{x}{p}\right] \log p = x \log x + O(x)$, where the sum is extended overall primes $\leq x$.

18. (a) Prove that two lattice point (a, b) and (m, n) are mutually visible if and only if $a - m$ and $b - n$ are relatively prime.

(b) Prove that the set of lattice points visible from the origin has density $6/\pi^2$.

19. (a) For any arithmetical function $a(n)$ let $A(x) = \sum_{n \leq x} a(n)$, where $A(x) = 0$ if $x < 1$. Assume f has continuous derivative on the interval $[y, x]$, where $0 < y < x$. Prove that :

$$\sum_{y < n \leq x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t) \, dt.$$

(b) Prove that for $n \geq 1$, the n^{th} prime p_n satisfies $\frac{1}{6} n \log n < p_n < 12\left(n \log n + n \log \frac{12}{e}\right)$.

20. (a) Assume $(a, m) = d$ and d/b. Prove that the linear congruence $ax \equiv (\bmod\, m)$ has exactly d solutions modulo m. Also obtain these solutions.

(b) Given a prime p, let $f(x) = c_0 + c_1 x + \ldots + c_n x^n$ be a polynomial of degree n with integer coefficients such that $c_n \neq O(\bmod\, p)$. Prove that the polynomial congruence $f(x) \equiv O(\bmod\, p)$ has atmost n solutions.

21. Assume $\alpha \geq 2$ and let r be a solution of the congruence $f(x) \equiv O\left(\bmod\, p^{\alpha-1}\right)$ lying in the interval $0 \leq r < p^{\alpha-1}$.

Prove the following :

(a) If $f^{'}(r) \equiv 0(\bmod\, p)$, then r can be lifted in a unique way from $p^{\alpha-1}$ to p^{α}.

(b) If $f^{'}(r) \equiv 0(\bmod\, p)$ (i) prove that when $f(r) \equiv O\left(\bmod\, p^{\alpha}\right)$, r can be lifted from $p^{\alpha-1}$ to p^{α} in p distinct ways.

(c) If $f(r) \not\equiv O\left(\bmod\, p^{\alpha}\right)$, then r cannot be lifted from $p^{\alpha-1}$ to p^{α}.

22. Prove that for $|x| < 1$, $\prod_{m=1}^{\infty} \frac{1}{1-x^m} = \sum_{n=0}^{\infty} p(n) x^n$, where $p(0) = 1$, $p(n)$ denote the partition function.

(3 × 5 = 15)