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Reg. No...............................

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M.Sc. DEGREE (C.S.S.) EXAMINATION, NOVEMBER 2021

Fourth Semester

Faculty of Science

Branch I (A)–Mathematics

MT 04 C 16—SPECTRAL THEORY

(Programme – Core – Common for all)

[2012 to 2018 Admissions—Supplementary/Mercy Chance]

Time : Three Hours — Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. Define weak convergence in a normed space. Let (x_n) be a sequence in a normed space X. Then prove that $\alpha x_n \xrightarrow{w} \alpha x$ where α is a scalar.

2. Let $X = C[0,1|$ and define $T : \mathscr{D}(T) \to X$ by $Tx = x'$, where the prime denotes differentiation and $\mathscr{D}(T)$ is the subspace of functions $x \in X$ which have a continuous derivative. Prove that T is not bounded but is closed.

3. Consider the Hilbert sequence space l^2. Define a linear operator $T : l^2 \to l^2$ by $T(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots)$ where $(\xi_1, \xi_2, \ldots) \in l^2$. Prove that 0 is a spectral value of T but 0 is not an eigenvalue of T.

4. Let X be a complex Banach space. Let T be a bounded linear operator from X to X. Prove that $\sigma(T)$ is compact.

5. Let X and Y be normed spaces. Let $T : X \to Y$ be a compact linear operator. Prove that αT is compact, where α is any scalar.

6. Prove that a densely defined linear operator T in a complex Hilbert space H is symmetric if and only if $T \subset T^x$.

Turn over

7. Prove that the spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert space H lies in the closed interval $[m, M]$ on the real axis, where $m = \inf_{\|x\|=1} \langle Tx, x \rangle, M = \inf_{\|x\|=1} \langle Tx, x \rangle.$

8. Let P_1 and P_2 be projections of a Hilbert space H onto Y_1 and Y_2, respectively, and $P_1\ P_2 = P_2P_1$ Show that $P_1 + P_2 - P_1P_2$ is a projection of H onto $Y_1 + Y_2$.

(5 × 1 = 5)

Part B

Answer any **five** *questions.*
Each question has weight 2.

9. Let (T_n) be a sequence of operators in B (X, Y), where X and Y are Banach spaces. Then prove that (T_n) is strongly operator convergent if and only if :

   (i) The sequence $(\| T_n \|)$ is bounded.

   (ii) The sequence $(T_n x)$ is Cauchy in Y for every x in a total subset M of X.

10. Let $T : \mathscr{D}(T) \to Y$ be a bounded linear operator with domain $\mathscr{D}(T) \subset X$, where X and Y are normed spaces. Prove that :

    (i) If $\mathscr{D}(T)$ is a closed subset of X, then T is closed.

    (ii) If T is closed and Y is complete, then $\mathscr{D}(T)$ is a closed subset of X.

11. State and prove Bounded inverse theorem.

12. Prove that the resolvent set $\rho(T)$ of a bounded linear operator T on a complex Banach space is open.

13. Let A be a complex Banach algebra with identity e. Let $x \in A$ and $\| x \| < 1$. Prove that $e - x$ is invertible and $(e - x)^{-1} = e + \sum_{j=1}^{\infty} x^3$.

14. Let A be a complex Banach algebra with identity e. Then for any $x \in A$, prove that it's spectrum $\sigma(x)$ is compact.

15. Let $T : X \to X$ be a compact linear operator on a normed space X, and let r be the smallest integer (depending on λ) such that $\mathcal{N}\left(T_\lambda^r\right) = \mathcal{N}\left(T_\lambda\right)^{r+1}$. Prove that $X = \mathcal{N}\left(T_\lambda^r\right) \in T_\lambda^r(X)$.

16. Let $P : H \to H$ be a bounded linear operator on a Hilbert space H. Prove that P is a projection if and only if P is self-adjoint and idempotent.

(5 × 2 = 10)

Part C

Answer any **three** *questions.*
Each question has weight 5.

17. State and prove open mapping theorem.

18. Let X be a complex Banach space and $T \in B(X, X)$. Let $r_\sigma(T)$ be spectral radius of T. Then prove that $r_\sigma(T) = \lim_{n \to \infty} \sqrt[n]{\| T^n \|}$.

19. Let $T : X \to Y$ be a compact linear operator. Prove that its adjoint operator $T^x : Y' \to X'$ is a compact linear operator, where X and Y are normed spaces and X' and Y' are dual spaces of X and Y.

20. (i) Prove that the set of eigenvalues of a compact linear operator $T : X \to X$ on a normed space X is countable and the only possible point of accumulation is $\lambda = 0$.

    (ii) Let $T : X \to X$ be a compact linear operator and $S : X \to X$ a bounded linear operator on a normed space. Then prove that ST and TS are compact.

21. If two bounded self-adjoint linear operators S and T on a Hilbert space H are positive and commute then prove that their product ST is positive.

22. Let H be a Hilbert space. Let P_1, P_2 be projections on H. Prove that :

    (i) $P = P_1P_2$ is a projection H if and only if the projections P_1 and P_2 commutes. Then P projects H onto $Y = Y_1 \cap Y_2$, where $Y_i = P_i(H), i = 1, 2$.

    (ii) Two closed subspaces Y and V of H are orthogonal if and only if the corresponding projections satisfy $P_Y P_V = 0$.

    (iii) The sum $P = P_1 + P_2$ is a projection on H if and only if $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ are orthogonal.

(3 × 5 = 15)