QP CODE: 21002034 21002034

Reg No : ....

Name : ....

M Sc DEGREE (CSS) EXAMINATION, NOVEMBER 2021

First Semester

CORE - ME010102 - LINEAR ALGEBRA

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 ADMISSION ONWARDS

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Time: 3 Hours Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define (x, y) + (x_1, y_1) = (x + x_1, y + y_1) and *c(x, y) = (cx, y). Is V, with these operations, a vector space over the field of real numbers?*

2. *Is the vector (3, −1, 0, −1) in the subspace of* $\mathbb{R}^4$ *spanned by the vectors (2, −1, 3, 2), (−1, 1, 1, −3), and (1, 1, 9, −5)?*

3. Show that the space C of complex numbers and R^2 are isomorphic, considering as vector spaces over R.

4. If $T : C^2 \to C^2$ is a linear operator defined as $T(x_1, x_2) = (x_1, 0)$,find the matrix of T in the standard ordered basis for C^2.

5. Prove that the transpose of a linear transformation is also linear.

6. If D is a 2-linear function with the property that D(A) = 0 for all 2 x 2 matrices A over K having equal rows, then show that D is alternating.

7. *Prove that the determinant of the Vandermonde matrix* $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ *is (b - a)(c - a) (c - b).*

8. *Write any 4 properties of the determinant function on* $K^{n \times n}$, *where K is a commutative ring with identity.*

9. *Find the characteristic values, if any, of the linear operator* T *on* $\mathbb{R}^2$ *which is represented in the standard ordered basis by the matrix* $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

10. *Let* W *be an invariant subspace for* T. *Prove that the characteristic polynomial of the restriction operator* T_W *divides the characteristic polynomial of* T.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. *Let V be the vector space of all 2 × 2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V which has*

four elements.

12. *Let* $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ *be the ordered basis for* $\mathbb{R}^3$ *consisting of* $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 1, 1)$, $\alpha_3 = (1, 0, 0)$. *What are the coordinates of the vector (a, b, c) in the ordered basis* $\mathcal{B}$*?*

13. Check whether the function $T : F^3 \to F^3$ defined as $T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$ is linear.

14. Let W be the subspace of R^4 spanned by the vectors $\alpha_1 = (1, 0, -1, 2)$ and $\alpha_2 = (2, 3, 1, 1)$ .Which linear functional $f(x_1, x_2, x_3, x_4) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ are in the annihilator of W.

15. Let f and g be linear functionals on a vector space V. If the null space of g contains the null space of f, prove that g is a scalar multiple of f.

16. *Define determinant function on* K^{nxn}*. Show that the function D from* K^{2x2} *to K defined by* $D(A)=A_{11}A_{22}-A_{12}A_{21}$ *is a determinant function.*

17. *Let* T *be a linear operator on* $\mathbb{R}^3$ *which is represented in the standard ordered basis by the matrix*
$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$
. Prove that T *is diagonalizable by exhibiting a basis for* $\mathbb{R}^3$ *, each vector of which is a characteristic vector of* T

18. *Let* A *be an* n×n *matrix. Prove that the characteristic and minimal polynomials for* A *have the same roots, except for multiplicities.*

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. Consider the 5 x 5 matrix A=
$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$
(a) Find an invertible matrix P such that PA is a row-reduced echelon matrix R.
(b) Find a basis for the row space W of A.
(c) Which vectors $(b_1, b_2, b_3, b_4, b_5)$ are in W?
(d) Find the coordinate matrix of each vector $(b_1, b_2, b_3, b_4, b_5)$ in W in the ordered basis chosen in (b).
(e) Write each vector $(b_1, b_2, b_3, b_4, b_5)$ in W as a linear combination of the rows of A.
(f) Give an explicit description of the vector space V of all 5 x 1 column matrices X such that AX = 0.
(g) Find a basis for V.

20. Let V and W be finite dimensional vector spaces over the field F such that dim V= dim W. If $T : V \to W$ be a linear transformation, prove that the following statements are equivalent:
    (i) T is invertible.
    (ii) T is non-singular.
    (iii) T is onto.
    (iv) If $\{\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n\}$ is a basis for V then $\{T\alpha_1, T\alpha_2, T\alpha_3, \ldots\ldots, T\alpha_n\}$ is a basis for W.

(v) There is some basis $\{\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n\}$ for V such that $\{T\alpha_1, T\alpha_2, T\alpha_3, \ldots\ldots, T\alpha_n\}$ is a basis for W

21. *If D is any alternating n-linear function on $K^{n \times n}$, then prove that for each n x n matrix A, D(A) =(det A)D(I).*

22. Let $V = W_1 \oplus \cdots \oplus W_k$, prove that there exist k linear operators $E_1, E_2, \cdots, E_k$ on V such that

i. Each E_i is a projection
ii. $E_i E_j = 0$ if $i \neq j$
iii. $I = E_1 + \cdots + E_k$
iv. The range of E_i is W_i

Also prove that if $E_1, E_2, \cdots, E_k$ are k linear operators on V which satisfy conditions (i), (ii) and (iii) and if we let W_i be the range of E_i then $V = W_1 \oplus \cdots \oplus W_k$

(2×5=10 weightage)