QP CODE: 21000686 21000686 **Reg No** : ....

Name : ....

M Sc DEGREE (CSS) EXAMINATION, JULY 2021

Fourth Semester

Faculty of Science

CORE - ME010401 - SPECTRAL THEORY

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 Admission Onwards

45B61C5D

Time: 3 Hours Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. When we can say that a subset of a metric space is rare, meager and nonmeager. State Baire's Category Theorem.
2. Define uniform,strong and weak operator convergence of a sequence of operators in $B(X,Y)$ where X and Y are normed spaces.
3. Define fixed point of a mapping defined on a set. Give an example of a mapping with two fixed points.
4. Define similar matrices. Prove that similar matrices have the same eigenvalues.
5. Define a domain and a holomorphic function in a complex plane C.
6. Let A be a complex Banach algebra with identity e. If for an element $x \in A$, there exist y and z in A such that $yx=e$ and $xz=e$ then show that x is invertible and $y=z=x^{-1}$.
7. Prove that the null space of ${T_{\lambda}}^{n}$ is finite dimensional for every $\lambda \neq 0$, where T is a compact linear operator on a normed space X.
8. Prove that every nonzero spectral value of a compact linear operator on a Banach space is an Eigen value.
9. Let $T:H\to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Prove that the residual spectrum $\sigma_r(T)$ is empty.
10. Define positive operators on a Hilbert space. Prove that the sum of two positive operators is positive.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Let Y be a proper closed subspace of a normed space X. Let $x_0 \in X-Y$ be arbitrary and $\delta = \inf\limits_{\tilde y \in Y} \Vert \tilde y -x_0 \Vert$, the distance from x_0 to Y. Prove that there exists an $\tilde f \in X'$ such that $\Vert \tilde f \Vert = 1$, $\tilde f (y) = 0$ for all $y \in Y$ and $\tilde f(x_0) = \delta$.
12. State and prove Closed Graph Theorem.
13. Prove that the spectrum $\sigma(T)$ of a bounded linear operator T on a complex Banach space X is compact and lies in the disc given by $|\lambda|\leq||T||$
14. Show that $\rho(T), \sigma_p(T), \sigma_c(T)$ and $\sigma_r(T)$ are mutually disjoint and their union is the complex plane.
15. Define compact linear operator on a normed space. Prove that every compact linear operator $T:X \to Y$, where X and Y are normed spaces is continuous.
16. Prove that the range of a compact linear operator $T:X\longrightarrow Y$ where X and Y are normed spaces, is separable.
17. Let $T:H\to H$ be a bounded linear operator on a complex Hilbert space H. Then prove that a number λ belongs to the resolvent set $\rho(T)$ of T if and only if there exists a $c>0$ such that for every $x\in H$, $\|T_{\lambda}x\|\ge c\|x\|$.
18. Define orthogonal projection on a Hilbert space. State and prove any four properties of an orthogonal projection.

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. Let T be a bounded linear operator from a Banach space X onto a Banach space Y. Prove that the image of the open unit ball $B_0=B(0,1)\subset X$ contains an open ball about $O\in Y$.
20. Let X be a complex Banach space, $T\in B(X,X)$ and $p(\lambda)=\alpha_{n}\lambda^n + \alpha_{n-1}\lambda^{n-1}+\cdots +\alpha_{0}$ is a polynomial of degree n , then prove that $\sigma (p(T))=p(\sigma (T))$.
21. Let A be a complex Banach algebra with identity e. Then show that $\sigma(x) \ne \phi$.
22. Let P_1 and P_2 be projections defined on a Hilbert space H and let $Y_1=P_1(H)$ and $Y_2=P_2(H)$.
    1. Prove that the difference $P=P_2-P_1$ is a projection if and only if $Y_1\subset Y_2$
    2. If $P=P_2-P_1$ is a projection, prove that P projects H onto $Y=Y_2\cap Y_1^\perp$.

(2×5=10 weightage)