QP CODE: 21000688

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Reg No : ....

Name : ....

M Sc DEGREE (CSS) EXAMINATION, JULY 2021

Fourth Semester

Faculty of Science

Elective - ME800401 - DIFFERENTIAL GEOMETRY

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 Admission Onwards

9B772118

Time: 3 Hours | Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Sketch the level set $f^{-1}(1)$ for the function $f(x_1, x_2) = x_1^2 + x_2^2$. Which points p of these level set fail to have tangent space equal to $[\nabla f(p)]^{\perp}$?
2. Define an oriented n-surface. Give an example.
3. Describe the spherical image, when $n = 2$, of $x_1^2 - x_2^2 - \ldots - x_{n+1}^2 = 4, x_1 > 0$ oriented by $\mathbf{N} = \frac{-\nabla f}{||\nabla f||}$.
4. Let $\mathbf{X}$ and $\mathbf{Y}$ be smooth vector fields along the parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$. Prove $(\mathbf{X} \cdot \mathbf{Y})' = \dot{\mathbf{X}} \cdot \mathbf{Y} + \mathbf{X} \cdot \dot{\mathbf{Y}}$.
5. Define Levi-Civita parallelism. Show that if $\mathbf{X}$ is a parallel vector field along α, then $\mathbf{X}$ has constant length.
6. Write a short note on the Weingarten map. Why is it called the shape operator of the surface.
7. Define length of the parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$. Find the length of the parametrized curve given by $\alpha(t) = (\sqrt{2}cos2t, sin2t, sin2t), I = [0, 2\pi], n = 2$.
8. Define an exact 1-form. Show that the integral of an exact 1-form over a compact connected oriented plane curve is always zero.
9. Let U be an open set in $\mathbb{R}^n$ and $\varphi : U \to \mathbb{R}^m$ be a smooth map.
   a) Define differential of φ.
   b) Show that $d\varphi(\mathbf{v})$ is independent of the choice of the parametrized curve.
10. a) Define coordinate vector fields along a smooth map $\varphi : U \to \mathbb{R}^{n+k}$, where Uopen in $\mathbb{R}^n$.
    b) Find the coordinate vector fields along φ of the parametrized torus φ in $\mathbb{R}^3$ given by
    $$\varphi(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi).$$

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Given the vector field $\mathbf{X}(p) = (p,\ \mathbf{X}(p))$ where $\mathbf{X}(p) = (0,1)$ then find the integral curve through an arbitrary point (a,b). Also if the curve passes through $(1,1)$ find the integral curve.

12. Show that the graph of a smooth real valued function on an open set U in $\mathbb{R}^n$ is an n-surface.

13. Show that if $\alpha : I \to S$ is a geodesic in an n-surface and if $\beta = \alpha \circ h$ is a reparametrization of α where $h : \tilde{I} \to I$ then β is a geodesic in S if and only if there exists $a, b \in \mathbb{R}$ with $a > 0$ such that $h(t) = at + b, \forall t \in \tilde{I}$.

14. Let S be a $2-$ surface in $\mathbb{R}^3$ and let $\alpha : I \to S$ be a geodesic in S with $\dot{\alpha} \neq 0$. Prove a vector field $\mathbf{X}$ tangent to S along α is parallel if and only if both $\|\mathbf{X}\|$ and the angle between $\mathbf{X}$ and $\dot{\alpha}$ are constant along α.

15. Are local parametrizations of plane curves unique upto reparametrization?Justify your answer.

16. State and prove Frenet formulas for a plane curve.

17. *Find the normal curvature of the sphere $x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = r^2$ of radius $r > 0$ oriented by the inward normal vector field.*

18. *Find the Gaussian curvature of the cone $x_1^2 + x_2^2 - x_3^2 = 0,\ x_3 > 0.$*

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. a) Define level set and graph of a function in $\mathbb{R}^{n+1}$. Also show that the graph of any function $f : \mathbb{R}^n \to \mathbb{R}$ is a level set for some function $F : \mathbb{R}^{n+1} \to \mathbb{R}$.
    b) Sketch typical level sets and graph of the function $f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 + x_2^2 + \ldots + x_{n+1}^2$. for $n = 0, 1$

20. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ exhibited as a level set $f^{-1}(c)$ of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\nabla f(p) \neq 0, \forall p \in S$. Prove that the Gauss map maps S onto the unit sphere S^n.

21. For the Weingarten map L_p, prove that $L_p(\mathbf{v}).\,\mathbf{w} = \mathbf{v}.\,L_p(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in S_p$.

22. *a) Prove that for each compact oriented n-surface S in $\mathbb{R}^{n+1}$ there exists a point p such that the second fundamental form at p is definite.*
    b) Let S be the ellipsoid $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$ oriented by the outward normal vector field. Find the Gauss - Kronecker curvature of S.

(2×5=10 weightage)