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M.Sc. DEGREE (C.S.S.) EXAMINATION, MAY 2020

Fourth Semester

Faculty of Science

Branch I (A)—Mathematics

MT04 C16—SPECTRAL THEORY

(2012 Admission onwards)

Time : Three Hours | Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. Suppose (x_n) is a sequence in a normed space X such that $x_n \xrightarrow{w} x$. Show that the weak limit x of (x_n) is unique.

2. Define Contraction on a metric space X. Prove that contraction on a metric space is continuous.

3. Consider the Hilbert sequence space l^2. Define a linear operator $T : l^2 \to l^2$ by $T(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots)$, where $(\xi_1, \xi_2, \ldots) \in l^2$. Prove that 0 is a spectral value of T but 0 is not an eigenvalue of T.

4. Show that for any operator $T \in B(X, X)$ on a complex Banach space X, $r_\sigma, (\alpha T) = \alpha r_\sigma (T)$.

5. Let X and Y be normed spaces. Let $T : X \to Y$ be a linear operator such that T maps every bounded sequence (x_n) in X onto a sequence (Tx_n) in Y has a convergent subsequence. Prove that T is compact.

6. Let X and Y be normed spaces. Let $T : X \to Y$ be a bounded linear operator with $\dim T(X) < \infty$. Prove that T is compact.

7. Prove that the spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert space H is real.

8. Let P_1 and P_2 be two projections on a Hilbert space H. Let $P_1 + P_2$ be a projection. Prove that $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ are orthogonal.

$(5 \times 1 = 5)$

Turn over

Part B

Answer any **five** *questions.*
Each question has weight 2.

9. Let (x_n) be a sequence in a normed space X. Prove that :

   (i) Strong convergence implies weak convergence with the same limit.

   (ii) If dim $X < \infty$, then weak convergence implies strong convergence.

10. Consider the space l^2. Let T_n be a sequence of operators from l^2 to l^2 defined by $T_n x = (\underbrace{0, 0, \ldots, 0}_{n \text{ zeros}}, \xi_1, \xi_2, \xi_3, \ldots)$ where $x = (\xi_1, \xi_2, \ldots) \in l^2$. Show that (T_n) is weakly operator convergent to 0 but not strongly.

11. State and prove Banach fixed point theorem.

12. Prove that all matrices representing a given linear operator $T : X \to X$ on a finite dimensional normed space X relative to various bases for X have the same eigen values.

13. Let A be a complex Banach algebra with identity. Let G be the set of all invertible elements of A. Then prove that G is an open subset of A and hence the subset $M = A - G$ of all non-invertible elements of A is closed.

14. Let X be a complex Banach space. Let $S, T \in B(X, X)$ then show that

   (i) $R_\mu - R_\lambda = (\mu - \lambda) R_\mu R_\lambda, \lambda, \mu \in \rho(T)$.

   (ii) $R_\lambda(S) - R_\lambda(T) = R_\lambda(S)(T - S) R_\lambda(T), \lambda \in \rho(S) \cap \rho(T)$.

15. Let (T_n) be a sequence of compact linear operators from a normed space X into a Banach space Y. If (T_n) is uniformly operator convergent to T, then prove that T is compact.

16. Let $T : H \to H$ be a bounded self adjoint linear operator on a complex Hilbert space H. Then prove that

   (a) all the eigenvalues of T (if they exists) are real.

   (b) all eigenvectors corresponding to different eigenvalues of T are orthogonal.

$(5 \times 2 = 10)$

Part C

Answer any **three** *questions.*
Each question has weight 5.

17. State and prove Open Mapping theorem.

18. Let A be a complex Banach algebra with identity e. Then for any $x \in A$ prove that

    (i) $\sigma(x)$ is compact.

    (ii) $\sigma(x) \neq \emptyset$.

19. Let $T : X \to X$ be a compact linear operator on a normed space X. Prove that for every $\lambda \neq 0$ the range of $T_\lambda = T - \lambda I$ is closed.

20. Let $S : \mathfrak{D}(S) \to H$ and $T : \mathfrak{D}(T) \to H$ be linear operators which are densely defined in a complex Hilbert space H. Then prove that

    (i) If $S \subset T$, then $T^* \subset S^*$.

    (ii) If $\mathfrak{D}(T^*)$ is dense in H, then $T \subset T^{**}$.

    (iii) If T is injective and its range $\mathfrak{R}(T)$ is dense in H, then T^* is injective and $(T^*)^{-1} = (T^{-1})^*$.

21. Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Prove that a number λ belongs to the resolvent set $\rho(T)$ of T if and only if there exists a $c > 0$ such that for every $x \in H$, $\| T_\lambda x \| \geq c \| x \|$, $(T_\lambda = T - \lambda I)$.

22. Define Monotone sequence. Let (T_n) be a sequence of hounded selfadjoint linear operators on a complex Hilbert space H such that $T_1 \leqq T_2 \leqq \ldots \leqq T_n \leqq \ldots \leqq K$ where K is a bounded self-adjoint linear operator on H. Suppose that any T_j commutes with K and with every T_m. Prove that (T_n) is strongly operator convergent ($T_n x \to T_x$ for all $x \in H$) and the limit operator T is linear, bounded and self-adjoint and satisfies $T \leqq K$.

$(3 \times 5 = 15)$