QP CODE: 20000646

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Reg No : ....

Name : ....

MSc DEGREE (CSS) EXAMINATION , NOVEMBER 2020

Second Semester

CORE - ME010205 - MEASURE AND INTEGRATION

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 Admission Onwards

3F75A89E

Time: 3 Hours | Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Define Lebesgue outer measure. Prove that the outer measure of the set of all rational numbers, $m^*(\mathbb{Q})$, is zero.
2. State and prove Borel-Cantelli Lemma
3. Is every measurable set Borel? Justify your answer.
4. Define a step function. Also define Riemann Integrablity of f over $[a, b]$, using integrals of step functions.
5. Prove that if a bounded function f defined on a closed bounded interval $[a, b]$ is Riemann integrable over $[a, b]$ then it is Lebesgue integrable over $[a, b]$ and the two integrals are equal.
6. Prove that, for an increasing sequence $\{f_n\}$ of nonnegative Lebesgue measureable functions on E, pointwise convergence a.e. of $\{f_n\}$ implies passage of limit under integral sign.
7. Define general outer measure and measurability.
8. Let $(X, \mathcal{M})$ be a measurable space where $\mathcal{M} = 2^X$. Which all are the functions that are measurable with respect to $\mathcal{M}$? Justify?
9. Comment on the positive part and negative part of a measurable function on a measurable space. When we say that the measurable function is integrable on X?
10. State Fubini's Theorem.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. If $\{E_k\}_{k=1}^n$ is any finite collection of Lebesgue measurable sets, then prove that $\bigcup_{k=1}^{n} E_k$ is Lebesgue measurable.

12. Let E be any set of real numbers. Then prove that E is measurable if and only if there is a G_δ-set G containing E for which $m^*(G - E) = 0$

13. Let f and g are Lebesgue measurable functions that are finite a.e. on E. Prove that the sum and product $f + g$ and fg are Lebesgue measurable on E.

14. State and prove Simple Approximation Lemma

15. State and prove the finite additivity, excision and countable monotonicity properties of general measure.

16. Prove that every measurable subset of a positive set is positive and the countable union of positive sets is positive.

17. Let $(X, \mathcal{M}, \mu)$ be a measure space and ν a finite measure on the measurable space $(X, \mathcal{M})$. State and prove any necessary and sufficient condition for ν to be absolutely continuous with respect to μ

18. Show that σ finiteness is necessary in the Radon Nikodym Theorem.

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19.
   1. State and prove Vitali's Theorem for non-measurable sets.
   2. Prove that there are disjoint sets of real numbers A and B for which $m^*(A \cup B) < m^*(A) + m^*(B)$.

20. Let f and g are nonnegative Lebesgue measurable functions defined E. Prove that
   1) For any α and β, $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$
   2) If $f \leq g$ on E, then $\int_E f \leq \int_E g$
   3) For disjoint subsets A and B of E , $\int_{AUB} f = \int_A f + \int_B f$

21. Let ν be a signed measure on the general measurable space $(X, \mathcal{M})$. Prove that there exists two unique pair of mutually singular measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$.

22. (a) State Radon Nikodym Theorem
   (b) Write a short note on Radon Nikodym Derivative.
   (c) State Lebesgue Decomposition theorem

(2×5=10 weightage)